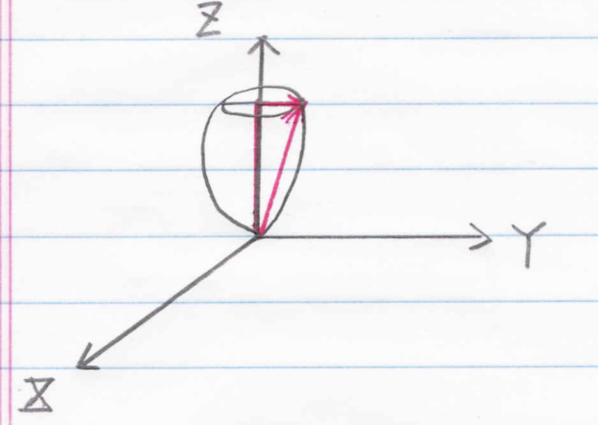
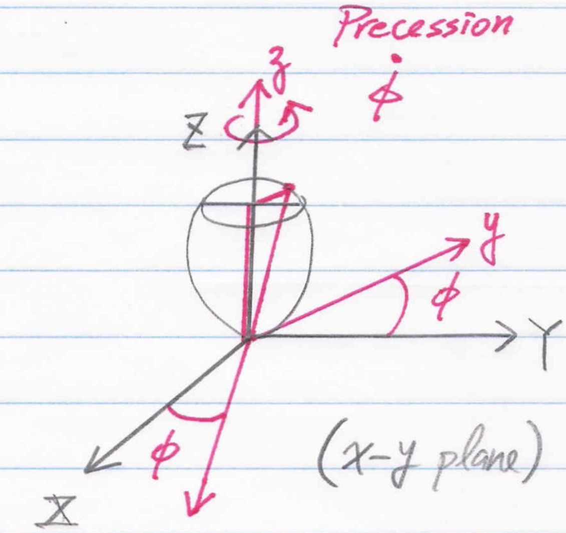
Gyroscopic Motion

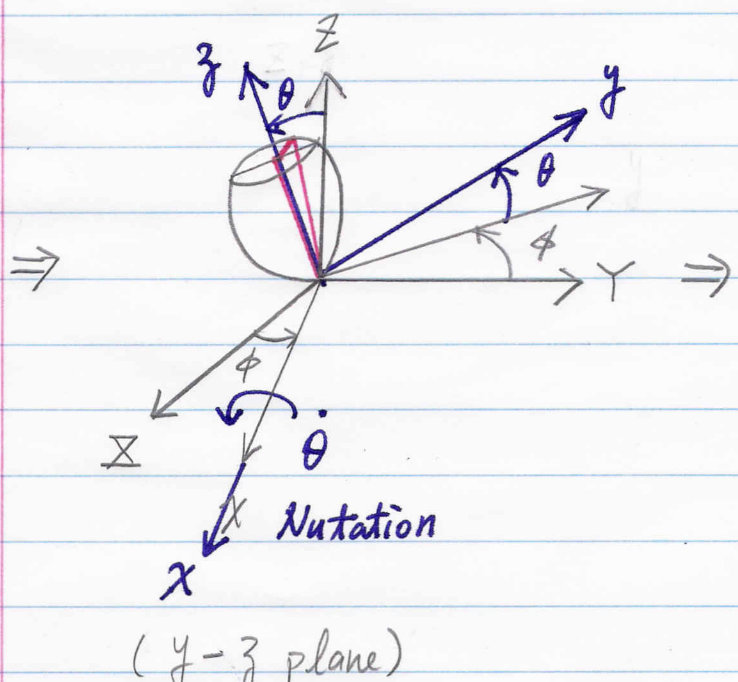
* Euler angles ϕ, θ, ψ



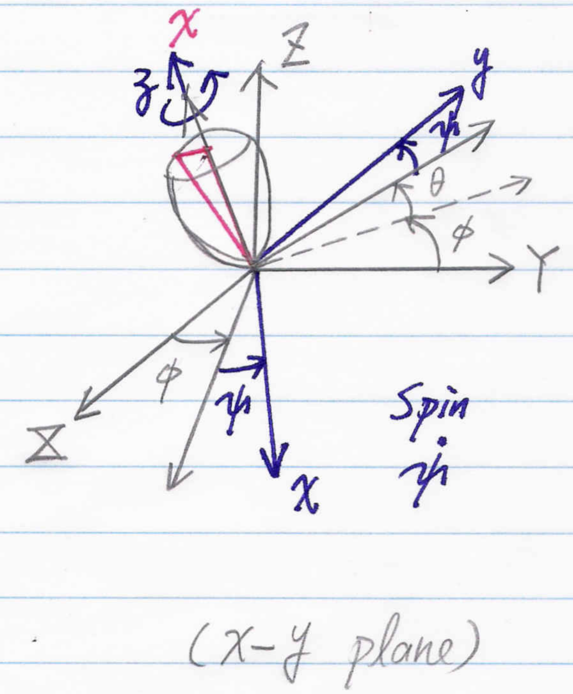
$\Rightarrow$



$\Rightarrow$



$\Rightarrow$

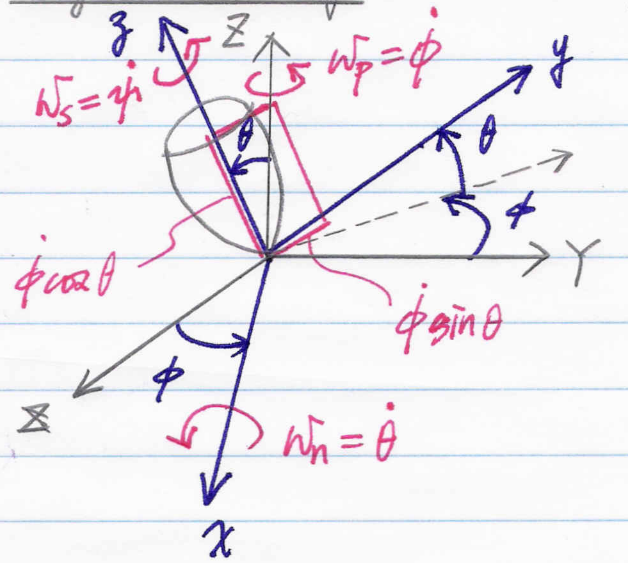


* ① Precession $\dot{\phi} = \dfrac{d\phi}{dt} = \dfrac{d}{dt}(\text{precession angle})$, $0 \le \phi \le 2\pi$

② Nutation $\dot{\theta} = \dfrac{d\theta}{dt} = \dfrac{d}{dt}(\text{nutation angle})$, $0 \le \theta \le \pi$

③ Spin $\dot{\psi} = \dfrac{d\psi}{dt} = \dfrac{d}{dt}(\text{spin angle})$, $0 \le \psi \le 2\pi$

* Angular velocity



* Rotation of the relative frame:

$$\vec{\Omega} = \vec{w_p} + \vec{w_n}$$

* Rotation of the body:

$$\vec{w} = w_x \vec{i} + w_y \vec{j} + w_z \vec{k}$$
$$= \dot{\theta}\,\vec{i} + (\dot{\phi}\sin\theta)\vec{j} + (\dot{\phi}\cos\theta + \dot{\psi})\vec{k} \;\#$$

$\because$ $\vec{w_p} = \dot{\phi}\sin\theta\,\vec{j} + \dot{\phi}\cos\theta\,\vec{k}$

$\vec{w_n} = \dot{\theta}\,\vec{i}$

$\vec{w_s} = \dot{\psi}\,\vec{k}$

$$\therefore \vec{\Omega} = \vec{w_p} + \vec{w_n} = \dot{\theta}\,\vec{i} + \dot{\phi}\sin\theta\,\vec{j} + \dot{\phi}\cos\theta\,\vec{k}$$
$$= \Omega_x \vec{i} + \Omega_y \vec{j} + \Omega_z \vec{k} \;\#$$

Substitute $\vec{w}$ & $\vec{\Omega}$ into Eq. (21-26).

$$\begin{cases} \sum M_x = I_x \dot{w}_x - I_y \Omega_z w_y + I_z \Omega_y w_z \\ \sum M_y = I_y \dot{w}_y - I_z \Omega_x w_z + I_x \Omega_z w_x \\ \sum M_z = I_z \dot{w}_z - I_x \Omega_y w_x + I_y \Omega_x w_y \end{cases}$$

* For a gyro, $I_{xx} = I_{yy} = I$ and $I_{zz} = I_z$.

We have

$$\begin{cases} \sum M_x = I\left(\ddot{\theta} - \dot{\phi}^2 \sin\theta \cos\theta\right) + I_z \dot{\phi} \sin\theta \underbrace{\left(\dot{\phi} \cos\theta + \dot{\psi}\right)}_{\omega_z} \\ \sum M_y = I\left(\ddot{\phi} \sin\theta + 2\dot{\phi}\dot{\theta} \cos\theta\right) - I_z \dot{\theta}\left(\dot{\phi} \cos\theta + \dot{\psi}\right) \\ \sum M_z = I_z\left(\ddot{\psi} + \ddot{\phi} \cos\theta - \dot{\phi}\dot{\theta} \sin\theta\right) \end{cases} \#$$

$$\Rightarrow \begin{cases} \sum M_x = f_x(\ddot{\theta}, \theta, \dot{\phi}, \phi, \dot{\psi}) \\ \sum M_y = f_y(\ddot{\phi}, \dot{\phi}, \dot{\theta}, \theta, \dot{\psi}) \\ \sum M_z = f_z(\ddot{\phi}, \dot{\phi}, \dot{\theta}, \theta, \ddot{\psi}) \end{cases}$$

* <u>Solution strategies</u>:

(1) Analytical solution: Closed-form; may exist when certain conditions are available.

(2) Numerical solution: Approximated by finite difference / finite element methods.

(3) Approximated analytical solution: Taylor series.

$\ast$ <u>Special case</u>: $\left(\theta = \text{const.} = \theta_c ,\ \dot{\phi} = \phi_c = \text{const.} ,\ \dot{\psi} = \dot{\psi}_c = \text{const.} \right)$

$\Rightarrow$ <u>Steady precession</u>

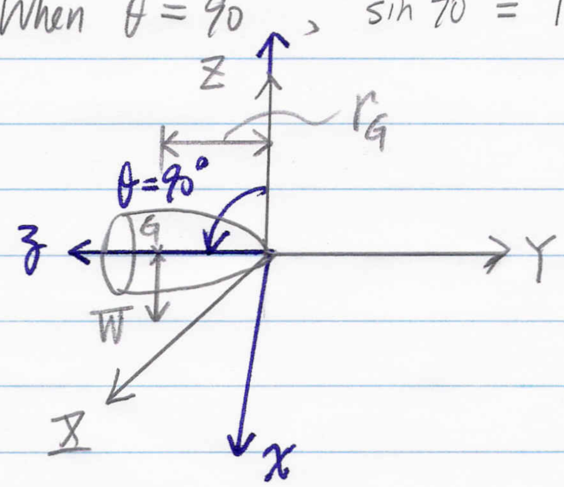
$\Rightarrow \dot{\theta} = \dot{\theta}_c = 0 ,\ \ddot{\phi} = \ddot{\phi}_c = 0 ,\ \ddot{\psi} = \ddot{\psi}_c = 0 \ ;\ \ddot{\theta} = \ddot{\theta}_c = 0$

Then we have

$$\Rightarrow \begin{cases} \sum M_x = I(0 - \dot{\phi}^2 \sin\theta\cos\theta) + I_z \dot{\phi} \sin\theta \underbrace{(\dot{\phi}\cos\theta + \dot{\psi})}_{\omega_z} \ \ast \\ \sum M_y = I(0+0) - I_z \times 0 \times (\dot{\phi}\cos\theta + \dot{\psi}) = 0 \\ \sum M_z = I_z(0 + 0 - 0) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \sum M_x = \dot{\phi}\sin\theta\left(I_z \omega_z - I\dot{\phi}\cos\theta\right) \\ \sum M_y = 0 \\ \sum M_z = 0 \ \ast \end{cases}$$

When $\theta = 90^\circ$, $\sin 90^\circ = 1$, $\cos 90^\circ = 0 \Rightarrow \sum M_x = I_z \omega_z \dot{\phi}$



Also, $\omega_z \Big|_{\theta=90^\circ} = \dot{\phi}\cos 90^\circ + \dot{\psi}$

$= \dot{\psi}$

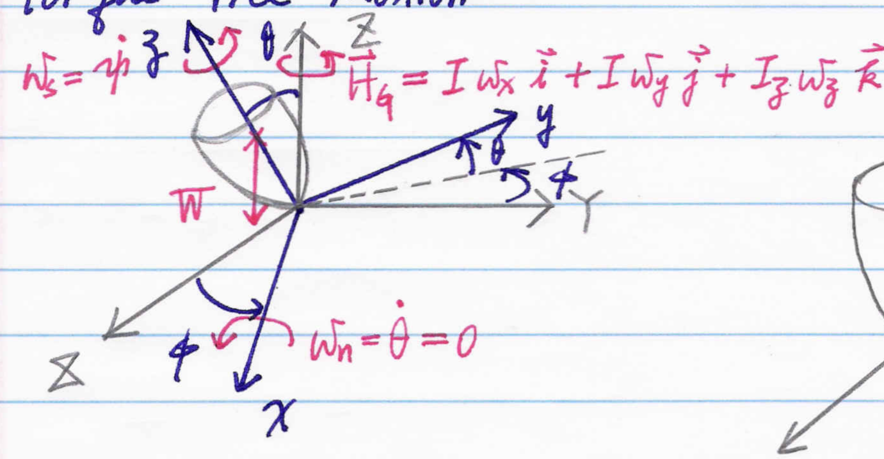
$\Omega_y \Big|_{\theta=90^\circ} = \dot{\phi}\sin 90^\circ = \dot{\phi}$

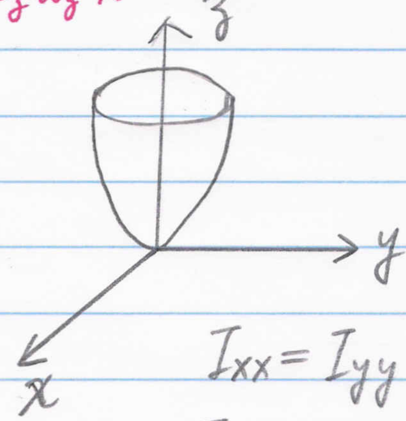
$\Rightarrow \boxed{\sum M_x = I_z \omega_z \Omega_y} \rightarrow$ Steady precession

$\Rightarrow \boxed{\sum M_x = I_z \omega_z \Omega_y = W \cdot r_G} \rightarrow$ The gyroscopic effect

$\boxed{\sum \vec{M}_x = \vec{\Omega}_y \times \vec{H}_o}$ where $\vec{\Omega}_y = \Omega_y \vec{j}$ & $\vec{H}_o = I_z \omega_z \vec{k}$ $\ast$

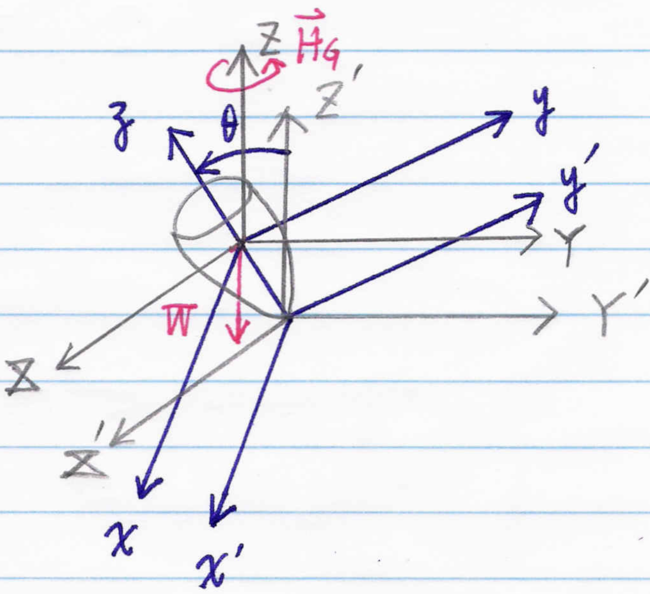
Torque-free Motion





$I_{xx} = I_{yy} = I$

$I_{zz} = I_z$



$\because$ Only gravity force

$\therefore \sum \vec{M}_G = \dot{\vec{H}}_G = 0$

$\Rightarrow \vec{H}_G = \text{const.}$

$\because \vec{H}_G$ is on y–z plane, $\therefore \vec{H}_G = H_G \cdot \sin\theta\, \vec{j} + H_G \cos\theta\, \vec{k}$

$$= I\omega_x \vec{i} + I\omega_y \vec{j} + I_z \omega_z \vec{k}$$

$\Rightarrow \because I \neq 0\ \&\ I_z \neq 0$

$\therefore \omega_x = 0,\quad \omega_y = \dfrac{H_G \sin\theta}{I},\quad \omega_z = \dfrac{H_G \cdot \cos\theta}{I_z}$

$$\Rightarrow \boxed{\vec{\omega} = 0\ \vec{i} + \frac{H_G \cdot \sin\theta}{I}\ \vec{j} + \frac{H_G \cos\theta}{I_z}\ \vec{k}}\ \#$$

$$= \dot{\theta}\ \vec{i} + \dot{\phi} \sin\theta\ \vec{j} + (\dot{\phi}\cos\theta + \dot{\psi})\ \vec{k}$$

Therefore,

$$\theta = \text{const.}$$

$$\dot{\phi}\sin\theta = \frac{H_G \sin\theta}{I} \Rightarrow \dot{\phi} = \frac{H_G}{I}$$

$$\dot{\phi}\cos\theta + \dot{\psi} = \frac{H_G\cos\theta}{I_z} \Rightarrow \dot{\psi} = \frac{I - I_z}{I I_z} H_G \cdot \cos\theta$$

//

$$\because H_G = \dot{\phi} I = \frac{I I_z}{I - I_z} \times \frac{\dot{\psi}}{\cos\theta}$$

$$\therefore \boxed{\dot{\psi} = \left(\frac{I - I_z}{I_z}\cos\theta\right)\dot{\phi}} \longrightarrow \text{Torque-free motion}$$

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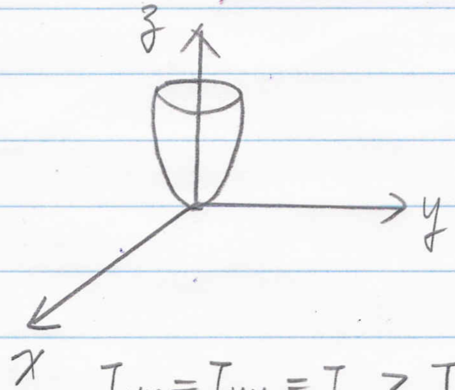
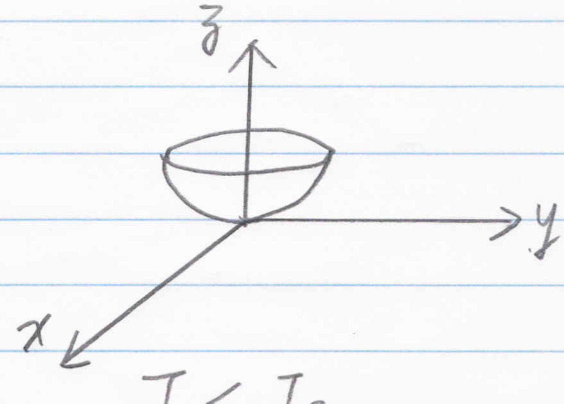
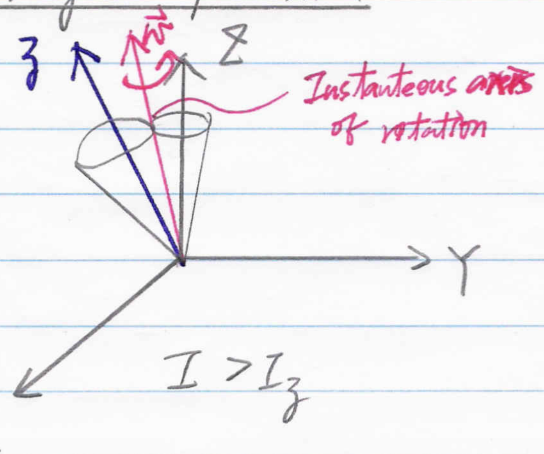


$I_{xx} = I_{yy} = I > I_{zz} = I_z$



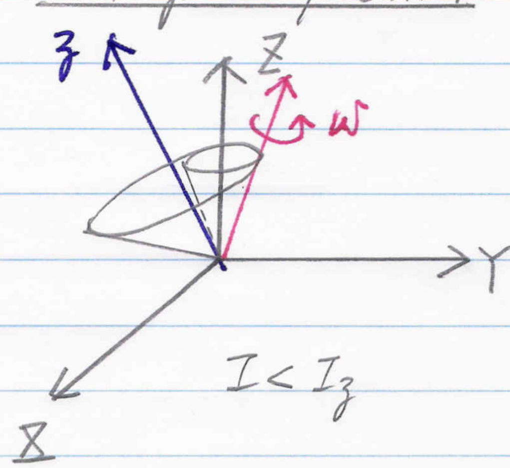
$I < I_z$

Regular precession



$I > I_z$

Retrograde precession



$I < I_z$