Forced Vibration of Single-Degree-of-Freedom (SDOF) Systems

- Dynamic response of SDOF systems subjected to external loading
  - Governing equation of motion
    \[ m\ddot{u} + c\dot{u} + ku = P(t) \] (1)
  
  the complete solution is
  \[ u = u_{\text{homogeneous}} + u_{\text{particular}} = u_h + u_p \] (2)

  where \( u_h \) is the homogeneous solution to the PDE or the free vibration response for \( P(t) = 0 \), and \( u_p \) is the particular solution to the PDE or the response for \( P(t) \neq 0 \).

- Types of motion (displacement) –
  1. Undamped systems \( (c = 0, \xi = 0) \) – Oscillation
  2. Undercritically-damped or underdamped systems \( (c < c_c, \xi < 1) \)
     - Oscillation (in general), also depending on I.C.
  3. Critically-damped systems \( (c = c_c, \xi = 1) \) – Decaying response with at most one reversal, depending on I.C.
  4. Overcritically-damped or overdamped systems \( (c > c_c, \xi > 1) \)
     - Decaying response, depending on I.C.

     * Case 1: Displacement never crosses the axis (non-reversal)
       \[ \left| \frac{u_0 + \xi \omega_n u_0}{\omega_D u_0} \right| < 1 \] (3)

     * Case 2: Displacement crosses the axis once
       \[ \left| \frac{u_0 + \xi \omega_n u_0}{\omega_D u_0} \right| > 1 \] (4)

where the critical damping coefficient \( c_c = 2\sqrt{km} \) (or \( c_{cr} \)) is the smallest value of \( c \) that exhibits oscillation completely. (See Figure 1.)
Figure 1: Free vibration displacement response ratio of SDOF systems

- **Particular solutions to the PDE with special loading functions**

  - **Impulse/Dirac delta function**

  The response of a SDOF system subjected to a unit impulse force having a finite time integral can be determined by the time integral for the force.

  \[
  \hat{P} = \int P(t)dt
  \]  

  (5)

  where \( \hat{P} \) is the linear impulse of force \( P(t) \). When the duration of \( P(t) \) approaches zero \( (t \to 0) \), the impulse force approaches infinity but \( \hat{P} \) becomes equal to unity or the unit impulse. This unit impulse is also known as the **Dirac delta function** defined by

  \[
  \delta(t - \tau) = 0
  \]  

  for \( t \neq \tau \) and

  \[
  \int_{0}^{\infty} \delta(t) dt = 1
  \]  

  (7)

  \[
  \int_{0}^{\infty} \delta(t - \tau) P(t) dt = P(\tau)
  \]  

  (8)

  in which \( 0 < \tau < \infty \). Therefore, an impulse force \( P(t) \) acting at \( t = \tau \) can be represented as

  \[
  P(t) = \hat{P} \delta(t - \tau)
  \]  

  (9)
Mathematically, \( t \to 0 \) is considered as \( t = t^+ \) (in real time). Eq.(8) becomes

\[
\int_0^{0^+} P(t)dt = \int_0^{0^+} \dot{P}\delta(t)dt = m\dot{u}(0^+) \tag{10}
\]

which is the impulse-momentum theorem. The initial velocity is

\[
\dot{u}(0^+) = \frac{\dot{P}}{m} \tag{11}
\]

The transient or free vibration displacement response for a SDOF system subjected to initial velocity becomes

\[
u(t) = \frac{\dot{P}}{m\omega_D} \sin(\omega_Dt) \exp(-\xi\omega_n t) \tag{12}
\]

where \( \omega_D = \omega_n \sqrt{1-\xi^2} \). Should we define

\[
u(t) = \dot{P}h(t), \tag{13}
\]

\( h(t) \) is the impulse response function and

\[
h(t) = \frac{1}{m\omega_D} \sin(\omega_Dt) \exp(-\xi\omega_n t) \tag{14}
\]

for damped SDOF systems. For undamped SDOF systems,

\[
h(t) = \frac{1}{m\omega_n} \sin(\omega_n t) \tag{15}
\]

– Duhamel’s integral –

The response of a SDOF system to arbitrary forms of excitation can be analyzed with the aid of the impulse function \( h(t) \) with magnitude of \( P(\tau) \). To do so, the arbitrary excitation \( P(t) \) is considered consisting of a sequence of impulse forces \( P(\tau) \) acting over a very small time interval \( d\tau \). The displacement response to each impulse is valid for all time \( t > \tau \). Therefore, the incremental response \( du \) to each impulse \( P(\tau) \) can be expressed as

\[
du = P(\tau)d\tau h(t - \tau) \tag{16}
\]
The total response to $P(t)$ is obtained by superimposing / integrating the individual incremental responses $du$ due to each impulse over the duration of loading.

$$u_p(t) = \int_0^t P(\tau)h(t-\tau)d\tau = P * h \quad (17)$$

Eq.(17) is known as Duhamel’s integral or the convolution integral, which is only applicable to linear systems. (Q: Why?) $*$ is the convolution symbol. Recall Eq.(14) for damped SDOF systems,

$$u_p(t) = \frac{1}{m\omega_D} \int_0^t P(\tau) \sin [\omega_D(t-\tau)] \exp \left[ -\xi\omega_n(t-\tau) \right] d\tau \quad (18)$$

The complete solution for impulse-loaded, undercritically-damped SDOF systems is obtained.

$$u(t) = u_h(t) + u_p(t)$$

$$= \left[ u_0 \cos \omega_D t + \left( \frac{\ddot{u}_0 + u_0\xi\omega_n}{\omega_D} \right) \sin \omega_D t \right] \cdot \exp (-\xi\omega_n t)$$

$$+ \frac{1}{m\omega_D} \int_0^t P(\tau) \sin [\omega_D(t-\tau)] \exp \left[ -\xi\omega_n(t-\tau) \right] d\tau \quad (19)$$

- Step load of infinite duration -

$$P(t) = \begin{cases} 0 & : t < 0 \\ P_0 & : t \geq 0 \end{cases} \quad (20)$$
– Forcing function is polynomial in time –

\[ P(t) = a + bt + ct^2 + \ldots \]  

(21)

where \( a, b, c, \ldots \) are constants in defining the forcing function.
• Harmonic vibration of SDOF systems
  
  – Undamped SDOF systems –

  \[ m\ddot{u} + ku = P(t) = P_0 \sin(\omega_p t) \quad (23) \]

  where \( \omega_p \) is the loading frequency. The particular solution is

  \[ u_p(t) = \frac{P_0}{k} \cdot \frac{1}{1 - \left(\frac{\omega_p}{\omega_n}\right)^2} \sin(\omega_p t) \quad (24) \]

  In harmonic vibration, due to the nature of external loading \( P(t) \), the complete solution represents the sum of two states:

  \[ u(t) = u_h(t) + u_p(t) \quad (25) \]

  where \( u_h(t) \) represents the transient state response and \( u_p(t) \) the steady state response. (See Figure 2.) The maximum static displacement is

  \[ (u_{st})_0 = \frac{P_0}{k} \quad (26) \]

  For \( \frac{\omega_p}{\omega_n} < 1 \) or \( \omega_p < \omega_n \), \( \frac{1}{1 - \left(\frac{\omega_p}{\omega_n}\right)^2} \) is positive, suggesting that \( u_p(t) \) and \( P(t) \) have same algebraic sign; the displacement is in
phase with the applied force. On the other hand, when \( \frac{\omega_p}{\omega_n} > 1 \) or \( \omega_p > \omega_n \), \( \frac{1}{1 - \left(\frac{\omega_p}{\omega_n}\right)^2} \) is negative and the displacement is out of phase with the applied force.

- Damped SDOF systems

\[
m\ddot{u} + c\dot{u} + ku = P(t) = P_0 \sin (\omega_p t)
\]
where \( \omega_p \) is the loading frequency. The particular solution is

\[
\dot{u}_p(t) = C \sin (\omega_p t) + D \cos (\omega_p t)
\]

where

\[
C = \frac{P_0}{k} \cdot \frac{1 - \left(\frac{\omega_p}{\omega_n}\right)^2}{\left[1 - \left(\frac{\omega_p}{\omega_n}\right)^2\right]^2 + \left[2\xi \left(\frac{\omega_p}{\omega_n}\right)\right]^2}
\]

\[
D = \frac{P_0}{k} \cdot \frac{-2\xi \frac{\omega_p}{\omega_n}}{\left[1 - \left(\frac{\omega_p}{\omega_n}\right)^2\right]^2 + \left[2\xi \left(\frac{\omega_p}{\omega_n}\right)\right]^2}
\]

The complete solution is still

\[
u(t) = u_h(t) + u_p(t)
\]
(See Figure 3.)

- Dynamic response factors

- Undamped SDOF systems

Eq.(24) can be written as

\[
\dot{u}_p(t) = (u_{st})_0 R_d \sin (\omega_p t - \phi)
\]
where the displacement response factor \( R_d \) and the phase angle (or phase lag) \( \phi \) are

\[
R_d = \frac{u_0}{(u_{st})_0} = \frac{1}{\left|1 - \left(\frac{\omega_p}{\omega_n}\right)^2\right|}
\]

\[
\phi = \begin{cases} 
0^\circ & : \omega_p < \omega_n \\
180^\circ & : \omega_p > \omega_n
\end{cases}
\]
In the case of $\omega_p = \omega_n$, $R_d \to \infty$; $\omega_p$ is the resonant frequency of the undamped SDOF system. However, resonance does not immediately result in excessive $R_d$ but only gradually lead to excessive $R_d$. Figure 4 shows the displacement ratio of an undamped SDOF system to sinusoidal force in resonance (loading frequency = natural frequency). Note the difference between Figure 2 and Figure 4. They are both harmonic response of an undamped SDOF system, except one is NOT in resonance (Figure 2) and the other is (4).

**Damped SDOF systems**

The displacement response factor $R_d$ and the phase angle $\phi$ for damped SDOF systems is

\[
R_d = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega_p}{\omega_n}\right)^2\right]^2 + 2\xi \left(\frac{\omega_p}{\omega_n}\right)^2}}
\]

\[
\phi = \tan^{-1} \left[ \frac{2\xi \left(\frac{\omega_p}{\omega_n}\right)}{1 - \left(\frac{\omega_p}{\omega_n}\right)^2} \right]
\]  

When $\frac{\omega_p}{\omega_n} \ll 1$, $R_d$ is independent of damping and

\[
u_0 \approx (u_{st})_0 = \frac{P_0}{k}
\]

which is the static deformation of the SDOF system. The dynamic response of this kind is controlled by the stiffness of the system. When the $\frac{\omega_p}{\omega_n} \gg 1$, $R_d$ approaches zero as $\frac{\omega_p}{\omega_n}$ increases and is unaffected by damping. If $\omega_p = \omega_n$,

\[
u_0 = \frac{(u_{st})_0}{2\xi} = \frac{P_0}{c\omega_n}
\]

And the phase angle $\phi$ is

\[
\phi = \left\{ \begin{array}{ll} 
0^\circ & : \omega_p \ll \omega_n \\
90^\circ & : \omega_p = \omega_n \\
180^\circ & : \omega_p \gg \omega_n 
\end{array} \right.
\]
The velocity response factor $R_v$ is
\[ R_v = \frac{\omega_p}{\omega_n} R_d \] (39)
and the acceleration response factor $R_a$ is
\[ R_a = \left(\frac{\omega_p}{\omega_n}\right)^2 R_d \] (40)

Or equivalently,
\[ \frac{R_a}{\omega_p} = \frac{\omega_v}{\omega_n} \] (41)

Figure 5 shows the frequency-response curves of the deformation response factor $R_d$ for a few values of $\xi$. Velocity and acceleration response factor curves are shown in Figure 6.

- **Natural frequency and damping from harmonic tests**
  - Resonance testing
    \[ \xi = \frac{1}{2} \frac{(U_{st})_0}{(U_0)_{\omega=\omega_n}} \] (42)
  - Frequency-response curve
    \[ \xi = \frac{\omega_b - \omega_n}{2\omega_n} = \frac{f_b - f_n}{2f_n} \] (43)

- **Force transmission and vibration isolation**
  - Transmissibility (TR): Figure 8 shows the transmissibility curves of various SDOF systems with different levels of damping.

- **Example – Transmissibility**

**Remark**

All figures in this lecture note are from Prof. Chopra’s book, *Dynamics of Structures*.

**Reading**

[AKC: Ch03 – 3.1, 3.2, Ch04 – 4.1~4.7]
Figure 2: Harmonic force and the displacement response of an undamped SDOF system; \( \frac{\omega}{\omega_n} = 0.2, \ u_0 = \frac{p_0}{2k} \)
Figure 3: Displacement response of an undercritically-damped SDOF system; \( \frac{\omega}{\omega_n} = 0.2, \xi = 0.05, u_0 = \frac{p_0}{2k} \)

Figure 4: Displacement response ratio of an undamped SDOF system; \( \frac{\omega}{\omega_n} = 1, \xi = 0, u_0 = \dot{u}_0 = 0, \)
Figure 5: Deformation response factor and phase angle for a damped system excited by harmonic force.
Figure 6: Deformation, velocity, and acceleration response factors for a damped system excited by harmonic force
Figure 7: Definition of half-power bandwidth

- $2\zeta = \text{Half-power bandwidth}$
- $(1/\sqrt{2}) \text{Resonant amplitude}$
Figure 8: Transmissibility of harmonic excitation