CIVE.5120 Structural Stability (3-0-3)
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Buckling of Rings, Curved Bars, and Arches

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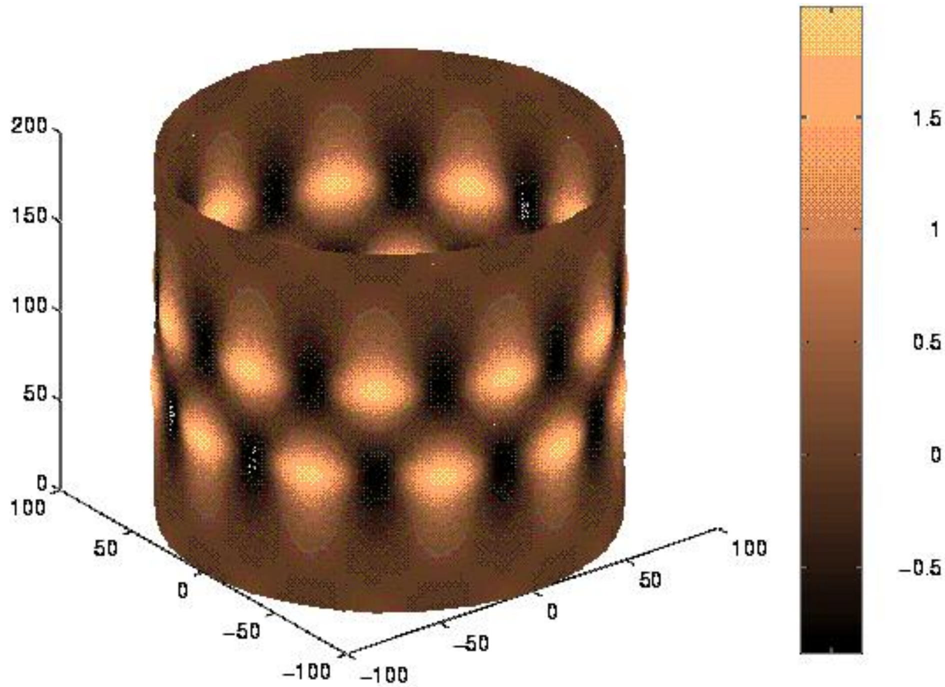
SERG

Outline

- Buckling failure of curved members
- Bending of a thin curved bar with a circular axis
- Condition of inextensional deformation of curved members
- Buckling of a circular ring under uniform pressure
- Arch action and types of arches
- Buckling of a uniformly loaded circular arch
  - Fixed-ended
  - Two-hinged
  - Three-hinged
- Buckling of a uniformly loaded parabolic arch
- Buckling of a uniformly loaded catenary/hyperbolic arch
- Summary
- References

Rings, Curved Bars, and Arches

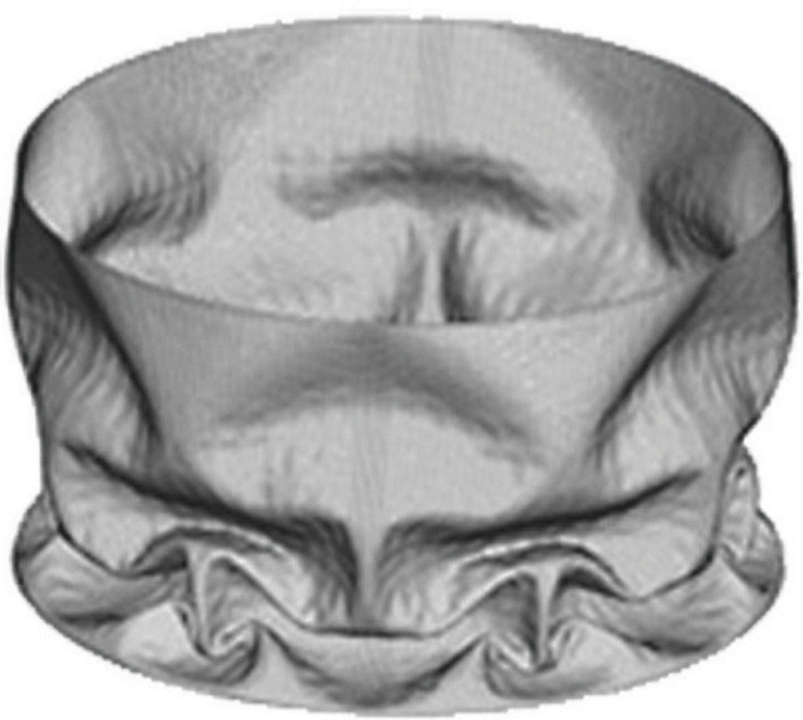
- **Buckling failure of curved members**



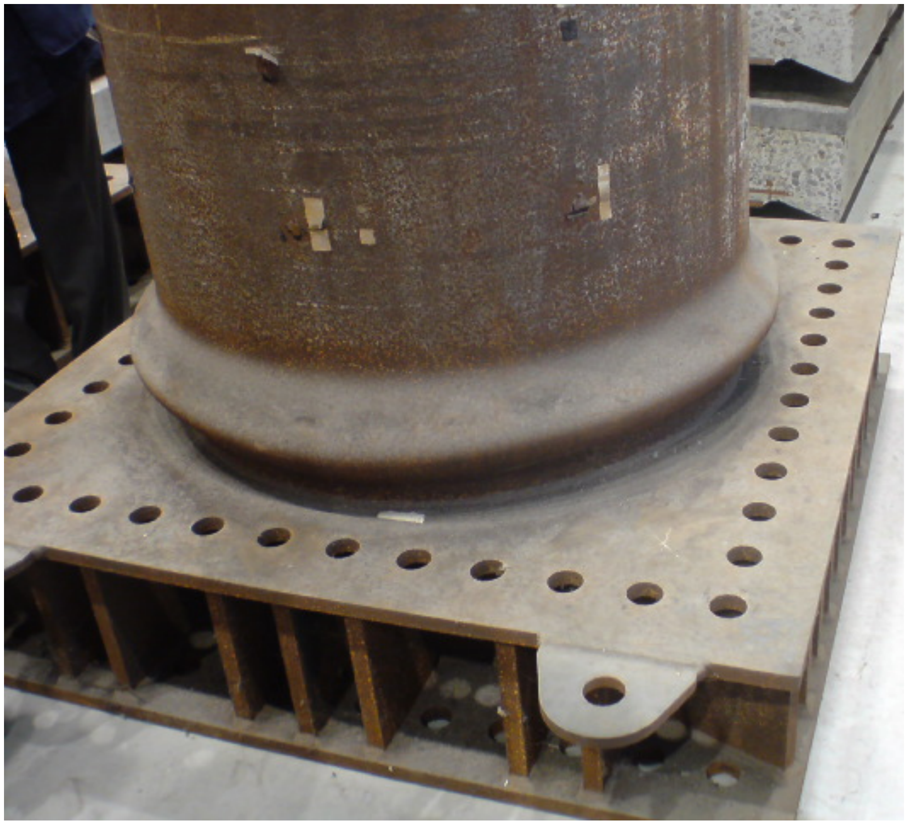
Buckling of an aluminum tube (Source: University of Bath, UK)

Rings, Curved Bars, and Arches

- **Buckling failure of tubes**

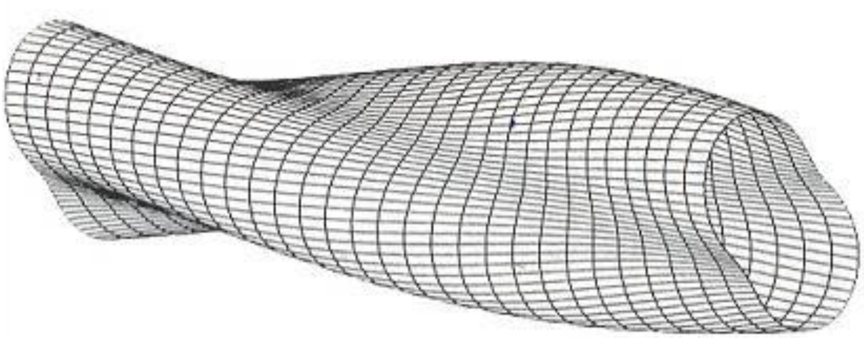


Simulated buckled tube (Source: NASA)



Local yielding at the bottom of a steel column, Public Works Research Institute (PWRI), Tsukuba, Japan (Source: T. Yu)

Rings, Curved Bars, and Arches

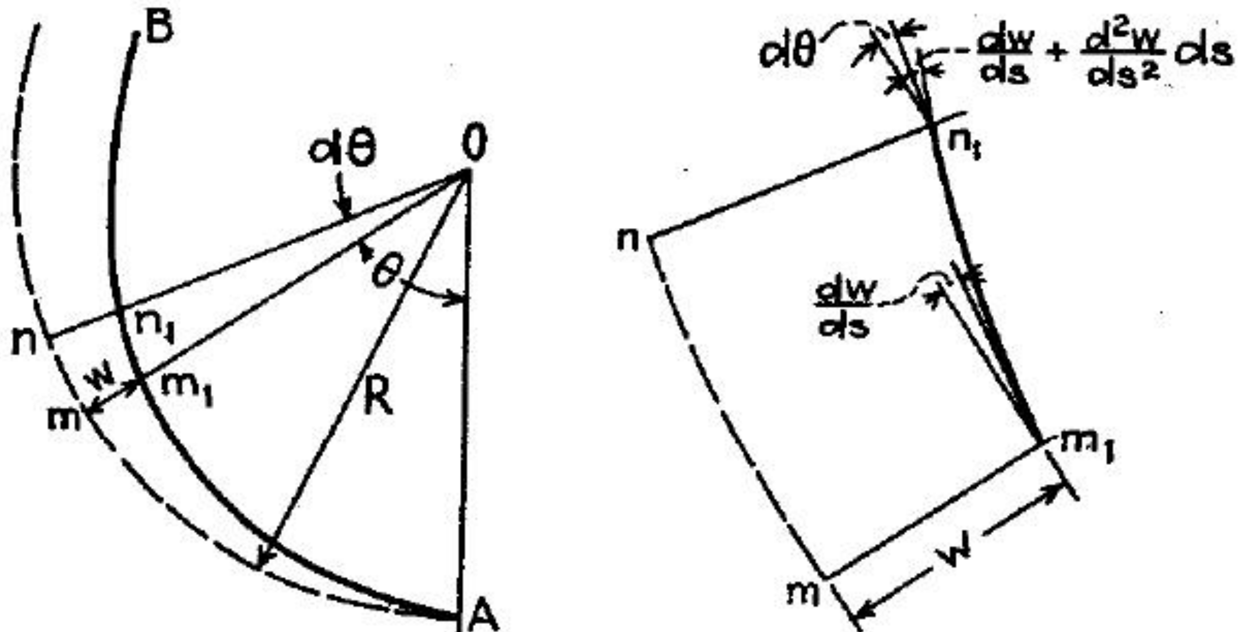


Torsional buckling of a tube
(Source: Wierzbicki, MIT)

Failure of a wind turbine tower
(Source: Greenward Technologies, Inc.)

Rings, Curved Bars, and Arches

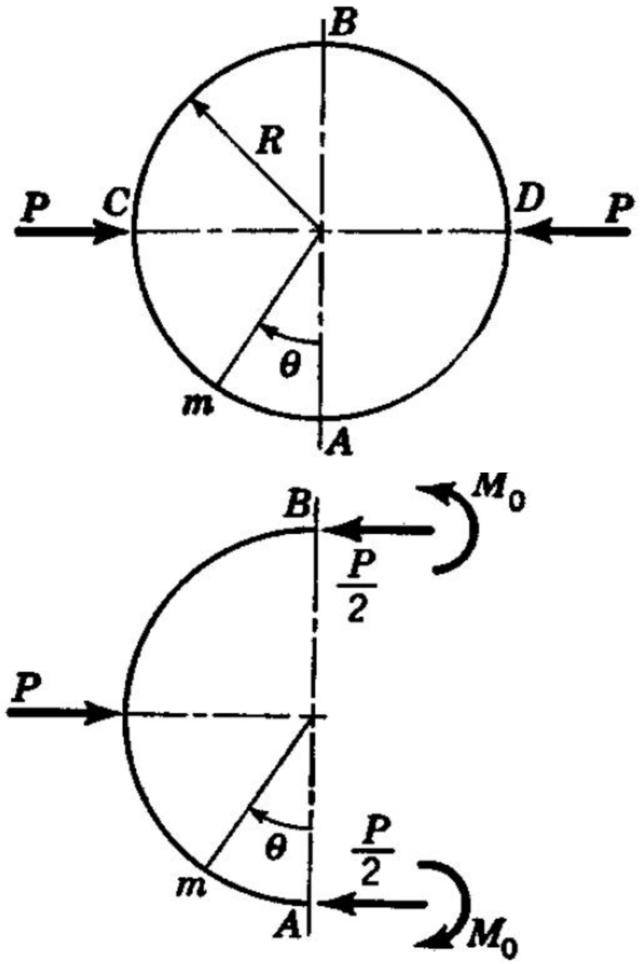
- **Bending of a thin curved bar with a circular axis**



(Source: Timoshenko and Gere 1961)

Rings, Curved Bars, and Arches

- **Bending of a thin curved bar with a circular axis**
  - Governing equation
  - Solution of the radial displacement, *w*



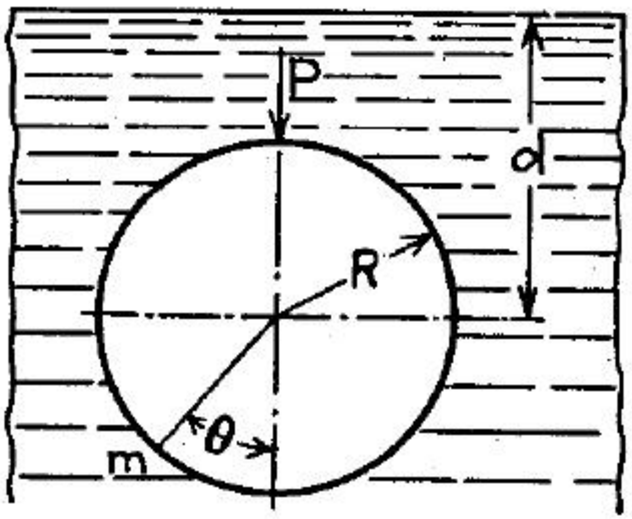
(Source: Timoshenko and Gere 1961)

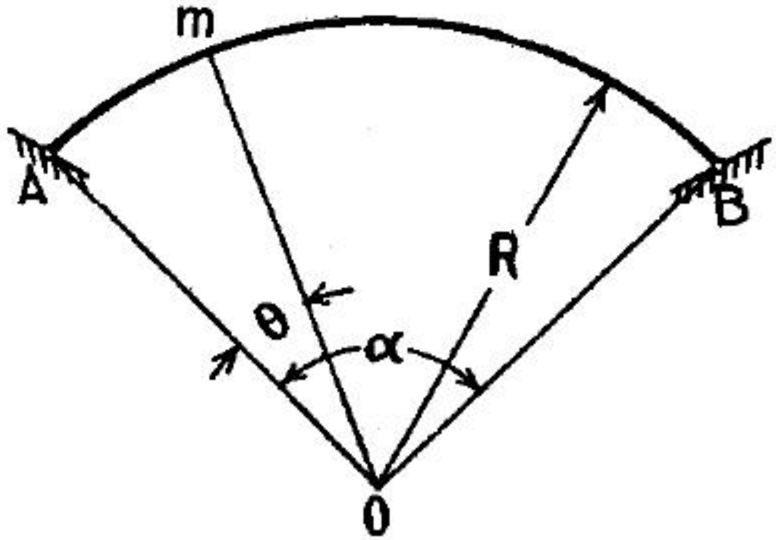
Rings, Curved Bars, and Arches

- **Condition of inextensional deformation of curved members**

Rings, Curved Bars, and Arches

- **Buckling of a circular ring under uniform pressure**

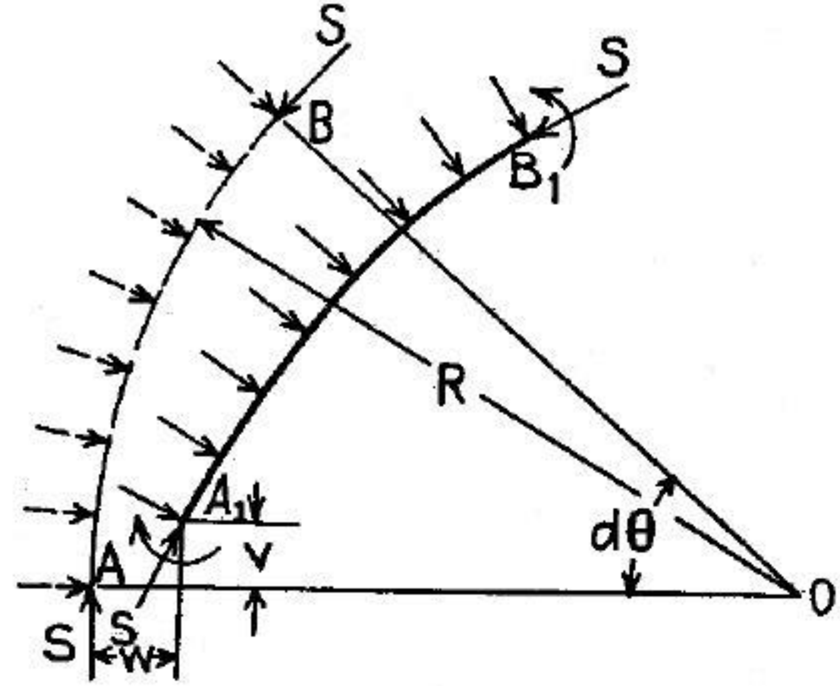




(Source: Timoshenko and Gere 1961)

Rings, Curved Bars, and Arches

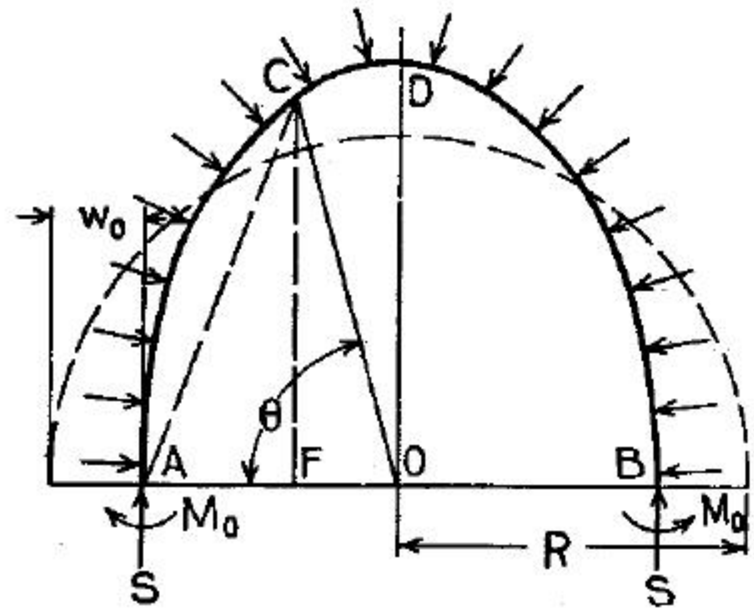
- **Buckling of a circular ring under uniform pressure**



(Source: Timoshenko and Gere 1961)

Rings, Curved Bars, and Arches

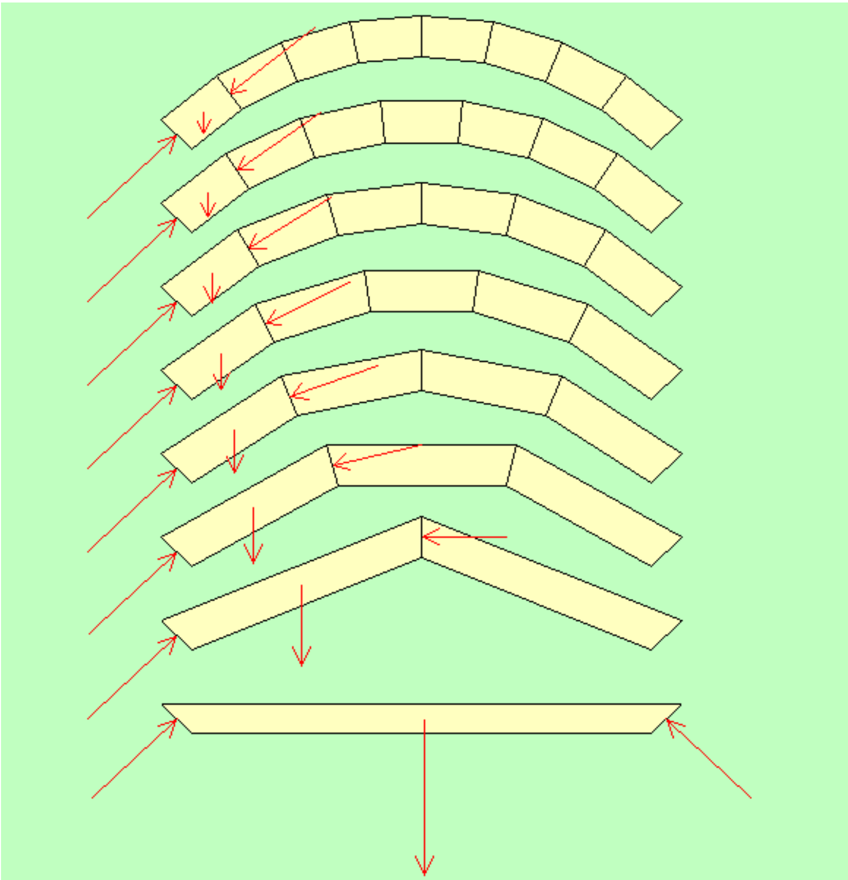
- **Buckling of a circular ring under uniform pressure**



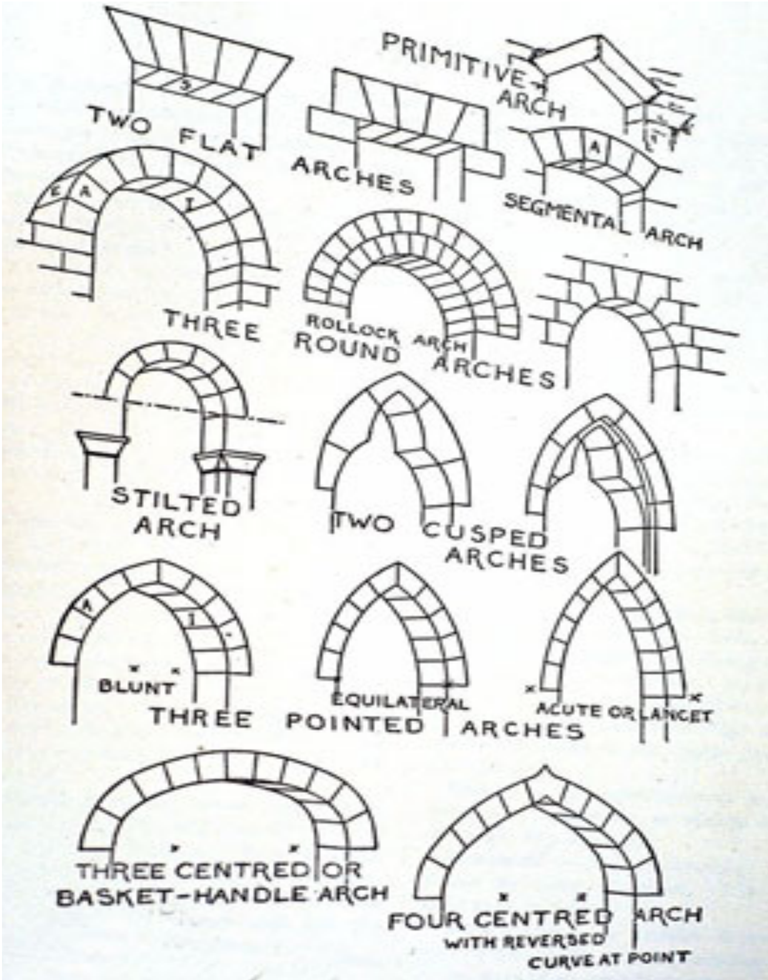
(Source: Timoshenko and Gere 1961)

Rings, Curved Bars, and Arches

- **Arch action**



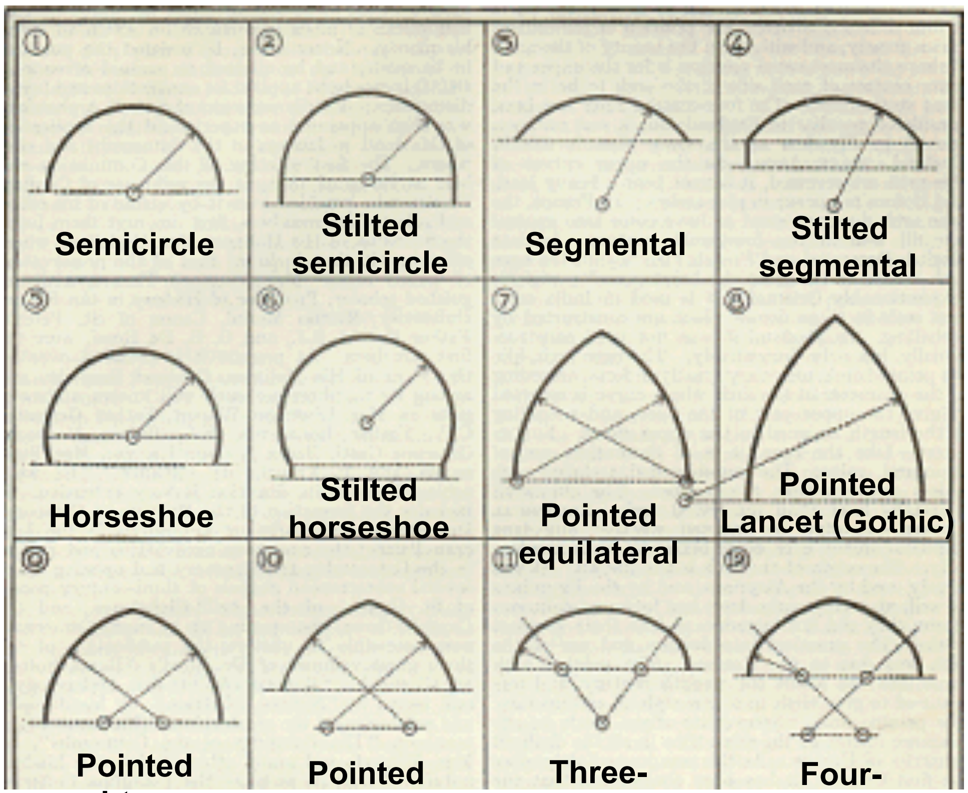
(Source: Asparis.Net)

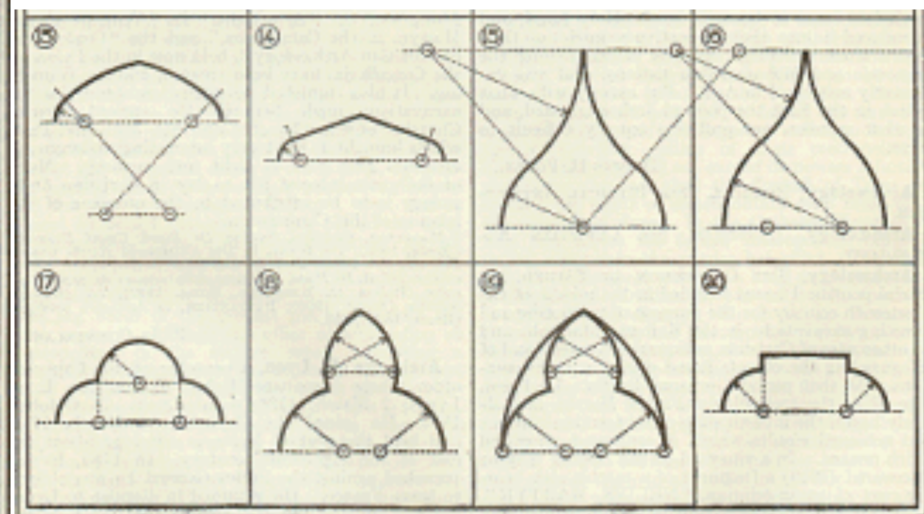


(Source: Sturgis Russell 1896)

Rings, Curved Bars, and Arches

- **Types of arches**

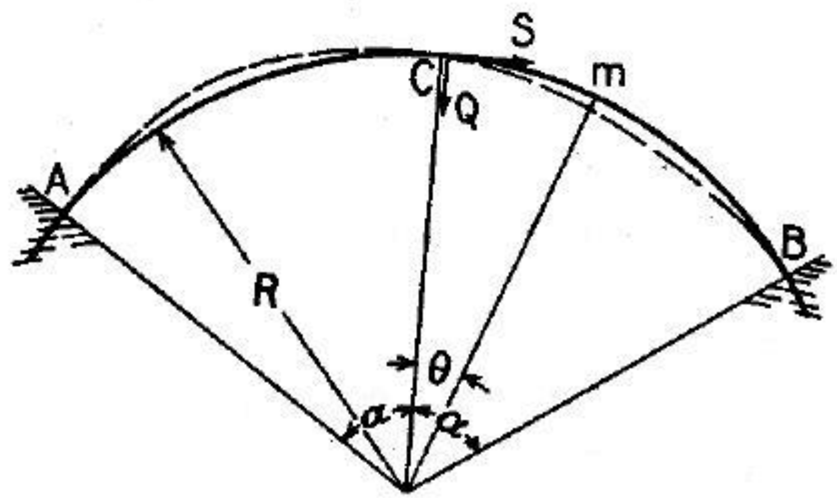




(Source: Catholicliturgy.com)

Rings, Curved Bars, and Arches

- **Buckling of a circular arch subjected to two point loads**
  - Fixed-ended



$$x^2 + y^2 = r^2$$

(Source: Timoshenko and Gere 1961)

Rings, Curved Bars, and Arches

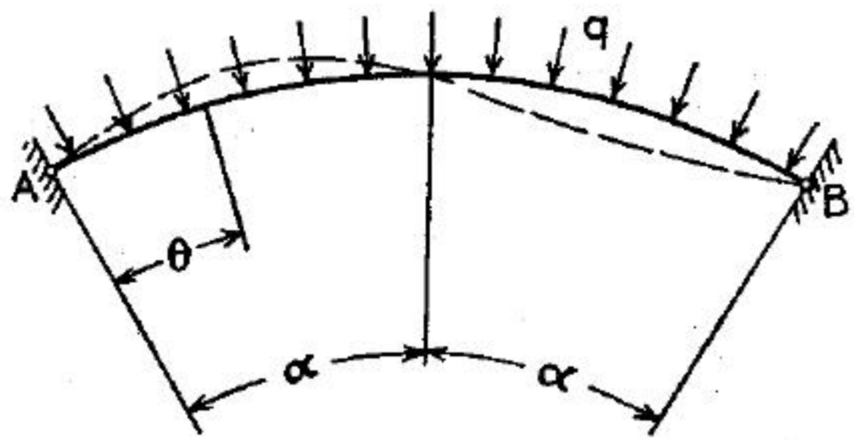
- **Buckling of a circular arch subjected to two point loads**
  - Fixed-ended

TABLE

α	30°	60°	90°	120°	150°	180°
k	8.621	4.375	3	2.364	2.066	2

Rings, Curved Bars, and Arches

- **Buckling of a uniformly loaded circular arch**
  - Two-hinged

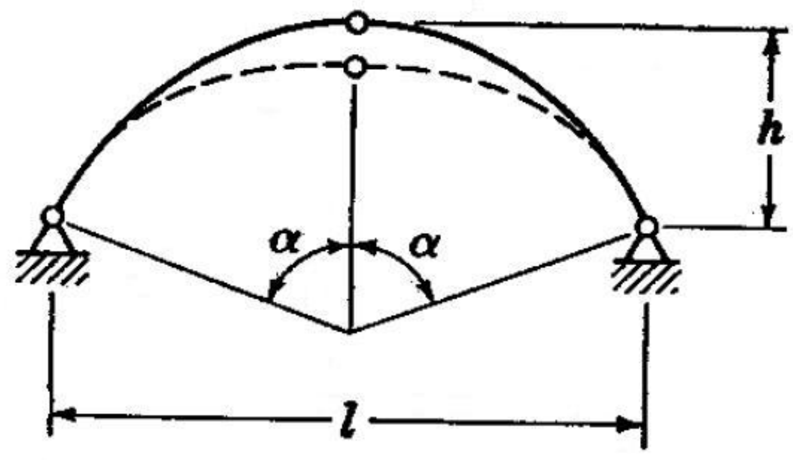


$$x^2 + y^2 = r^2$$

(Source: Timoshenko and Gere 1961)

Rings, Curved Bars, and Arches

- **Buckling of a uniformly loaded circular arch**
  - Three-hinged



$$x^2 + y^2 = r^2$$

(Source: Timoshenko and Gere 1961)

Rings, Curved Bars, and Arches

- **Buckling of a uniformly loaded circular arch**
  - Three-hinged

TABLE VALUES OF THE FACTOR γ_1 FOR UNIFORMLY COMPRESSED CIRCULAR ARCHES OF CONSTANT CROSS SECTION

2α (deg)	No hinges	One hinge	Two hinges	Three hinges
30	294	162	143	108
60	73.3	40.2	35	27.6
90	32.4	17.4	15	12.0
120	18.1	10.2	8	6.75
150	11.5	6.56	4.76	4.32
180	8.0	4.61	3.00	3.00

Rings, Curved Bars, and Arches

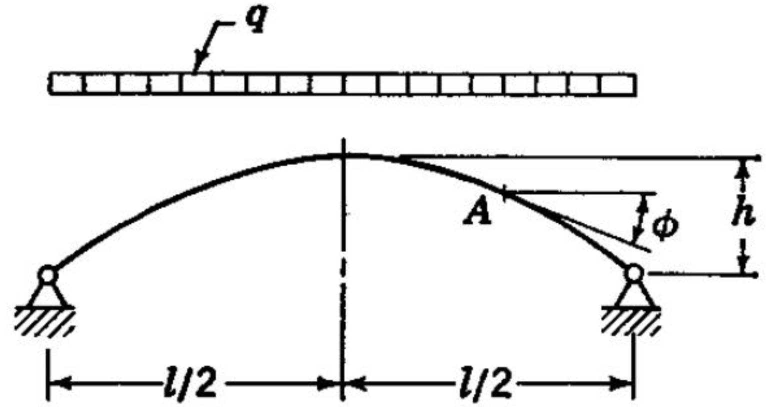
- **Buckling of circular arches – Comparison**

TABLE VALUES OF THE FACTOR γ_2 FOR UNIFORMLY COMPRESSED CIRCULAR ARCHES OF CONSTANT CROSS SECTION

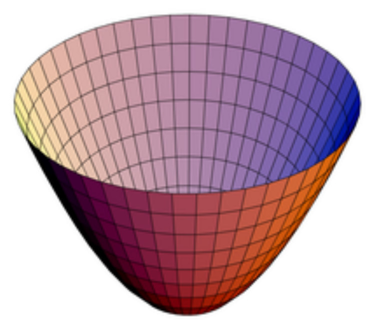
$\frac{h}{l}$	No hinges	One hinge	Two hinges	Three hinges
0.1	58.9	33	28.4	22.2
0.2	90.4	50	39.3	33.5
0.3	93.4	52	40.9	34.9
0.4	80.7	46	32.8	30.2
0.5	64.0	37	24.0	24.0

Rings, Curved Bars, and Arches

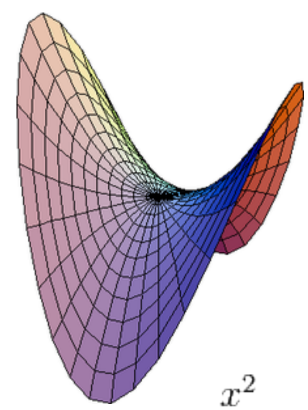
- **Buckling of a uniformly loaded parabolic arch**



(Source: Timoshenko and Gere 1961)



$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$



$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

Rings, Curved Bars, and Arches

- **Buckling of a uniformly loaded parabolic arch**

TABLE VALUES OF THE FACTOR γ_4 FOR PARABOLIC ARCHES OF CONSTANT CROSS SECTION WITH UNIFORM LOAD

$\frac{h}{l}$	No hinges	One hinge	Two hinges	Three hinges
0.1	60.7	33.8	28.5	22.5
0.2	101	59	45.4	39.6
0.3	115	.....	46.5	46.5
0.4	111	96	43.9	43.9
0.5	97.4	.....	38.4	38.4
0.6	83.8	80	30.5	30.5
0.8	59.1	59.1	20.0	20.0
1.0	43.7	43.7	14.1	14.1

(Source: Timoshenko and Gere 1961)

Rings, Curved Bars, and Arches

- **Buckling of a uniformly loaded parabolic arch**

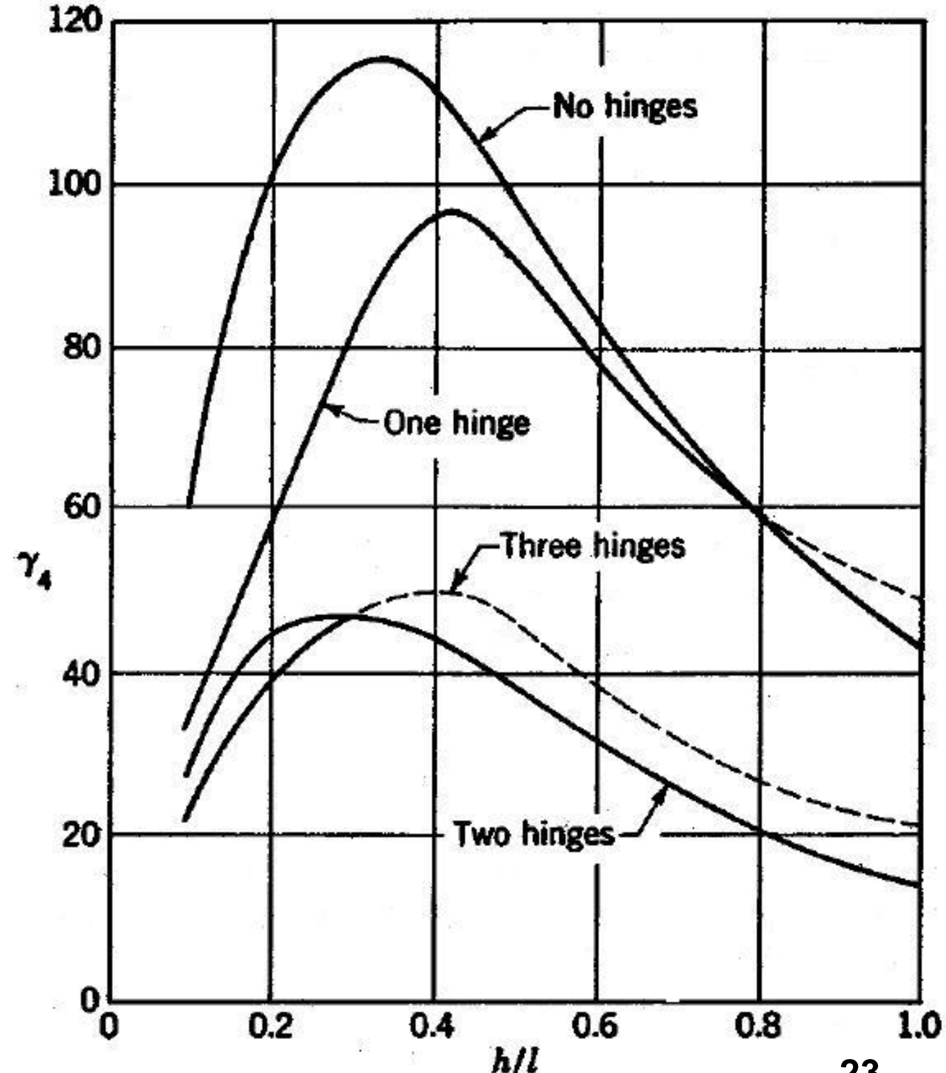
TABLE VALUES OF THE FACTOR γ_4 FOR PARABOLIC ARCHES OF VARYING CROSS SECTION WITH UNIFORM LOAD

$\frac{h}{l}$	No hinges	One hinge	Two hinges	Three hinges
0.1	65.5	36.5	30.7	24
0.2	134	75.8	59.8	51.2
0.3	204	......	81.1	81.1
0.4	277	187	101	101
0.6	444	332	142	142
0.8	587	497	170	170
1.0	700	697	193	193

(Source: Timoshenko and Gere 1961)

Rings, Curved Bars, and Arches

- **Buckling of a uniformly loaded parabolic arch**
  - **Effect of boundary condition**

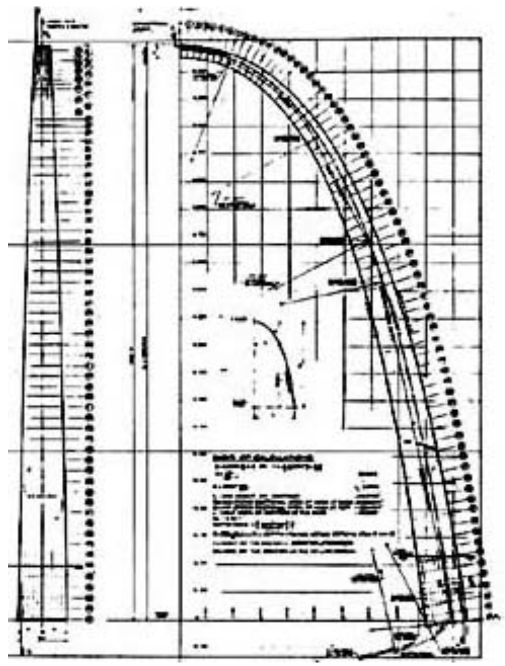


(Source: Timoshenko and Gere 1961)

Rings, Curved Bars, and Arches

- **Buckling of a uniformly loaded catenary/hyperbolic arch**

The St. Louis Gateway Arch, Missouri



$$Y = A\left(\cosh\frac{X}{L}C - 1\right)$$

$$X = \frac{L}{C}\left[\cosh^{-1}\left(1 + \frac{Y}{A}\right)\right]$$

$$\text{Where, } A = \frac{f_c}{\frac{Q_b}{Q_t} - 1} = 68.7672$$

$$C = \cosh^{-1}\frac{Q_b}{Q_t} = 3.0022$$

f_c = Max. Ht. of Centroid = 625.0925

Q_b = Max. X-Sec. @ Arch Base = 1262.6651

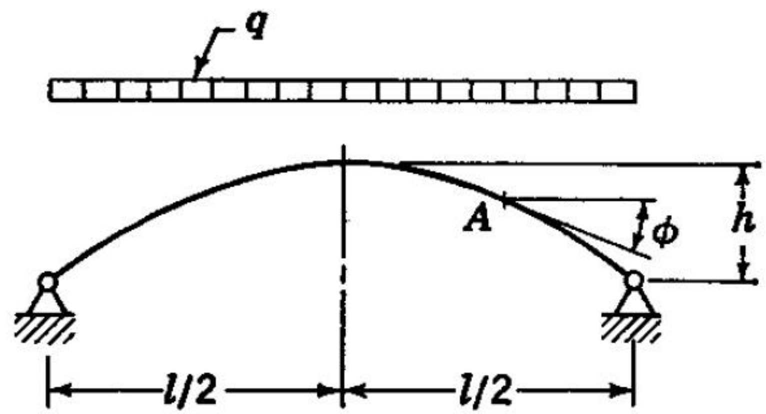
Q_t = Min. X-Sec. @ Arch Top = 125.1406

L = Half of Centroid @ Arch Base = 299.2239

→ Varying cross section; no internal bending moments

Rings, Curved Bars, and Arches

- **Buckling of a uniformly loaded catenary/hyperbolic arch**



$$y = a\cosh\left(\frac{x}{a}\right) = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$$

(Source: Timoshenko and Gere 1961)

Rings, Curved Bars, and Arches

- **Buckling of a uniformly loaded catenary/hyperbolic arch**

TABLE VALUES OF THE FACTOR γ_4 FOR CATENARY ARCHES OF CONSTANT CROSS SECTION WITH LOAD UNIFORMLY DISTRIBUTED ALONG THE ARCH AXIS

$\frac{h}{l}$	No hinges	Two hinges
0.1	59.4	28.4
0.2	96.4	43.2
0.3	112.0	41.9
0.4	92.3	35.4
0.5	80.7	27.4
1.0	27.8	7.06

(Source: Timoshenko and Gere 1961)

Summary

- The critical load of arches depends on i) arch shape (geometry and the aspect ratio), ii) cross-sectional properties, iii) boundary conditions, and iv) types of loading.

- For fixed-ended parabolic arches, the critical load reaches its maximum value when an optimal aspect ratio is found.

- For fixed-ended catenary arches, the critical load reaches its maximum value when the aspect ratio equals unity.

Summary

- In the analysis of curved members, the radial displacement in curved members is usually assumed to be small.
- The boundary condition of rings (plane stress) is different from the one of tubes (plane strain); this leads to the use of different expressions of Young’s modulus.
- The critical load of arches depends on i) arch shape (geometry and the aspect ratio), ii) cross-sectional properties, iii) boundary conditions, and iv) types of loading.
- For fixed-ended parabolic arches, the critical load reaches its maximum value when an optimal aspect ratio is found.
- For fixed-ended catenary arches, the critical load reaches its maximum value when the aspect ratio equals unity.

Reference

- S.P. Timoshenko, J.M. Gere (1961), *Theory of Elastic Stability*, 2nd ed., McGraw-Hill, New York, NY.