Buckling of Uniformly Loaded Circular Thin Plates

Consider a circular thin plate uniformly loaded with a force N_r (force per length) shown in Figure 1. From the free body diagram of an infinitesimal element, the governing equation can be developed as

$$r^2 \frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} - \phi = -\frac{N_r \phi r^2}{D} \tag{1}$$

where r = radial variable, ϕ = angular variable, $D = \dfrac{Et^3}{12(1-\nu^2)}$ = flexural rigidity of thin plates, t = plate thickness, and E = Young's modulus, ν = Poisson's ratio. Define

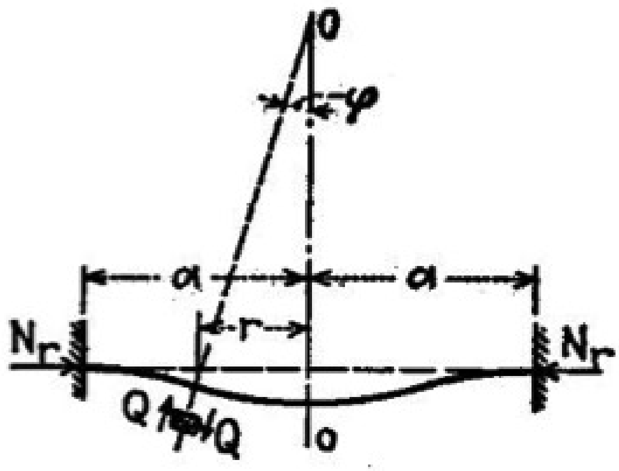

Figure 1: Definition of variables

$$\alpha^2 = \frac{N_r}{D} \tag{2}$$

Eq.(1) becomes

$$r^2 \frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} + \left(\alpha^2 r^2 - 1\right)\phi = 0 \tag{3}$$

Let $u = \alpha r$, we have

$$u^2 \frac{d^2\phi}{du^2} + u\frac{d\phi}{du} + \left(u^2 - 1\right)\phi = 0 \tag{4}$$

Boundary conditions provide

$$\phi(r = a) = 0 \tag{5}$$

$$\phi(r = 0) = 0 \tag{6}$$

where a = radius of the circular thin plate. Eq.(5) indicates fixed end for the circular thin plate. Eq.(6) suggests symmetric deformation profile.

The solution of Eq.(4) is

$$\phi(r) = A_1 J_1 \left(r\sqrt{\frac{N_r}{D}} \right) + A_2 Y_1 \left(r\sqrt{\frac{N_r}{D}} \right) \tag{7}$$

where A_1 and A_2 = constants, J_1 = Bessel function of the first kind, Y_1 = Bessel function of the second kind.

By finding the extreme values of $\phi(r)$, the critical load can be found to be

$$\underline{N_{cr} = (N_r)_{cr} = \frac{14.68D}{a^2}}. \tag{8}$$

For simply supported circular thin plates, boundary conditions become

$$\phi''(r = a) = 0 \tag{9}$$

$$\phi(r = 0) = 0 \tag{10}$$

Consequently, the critical load is

$$\underline{N_{cr} = (N_r)_{cr} = \frac{4.2D}{a^2}}. \tag{11}$$