

# CIVE 5120 Structural Stability

LN12

## Advanced Topics

#1

### ① Application of the Lyapunov Stability Theorems

\* Example #5 (Dynamical Stability): The van der Pol equation

① Governing equation:

$$\ddot{y} + y - \varepsilon \left( \frac{\dot{y}^3}{3} - y \right) = 0$$

② Let  $x_1 = y$ ,  $\dot{x}_1 = \frac{dx_1}{dt} = x_2$

$$\Rightarrow \dot{x}_2 = -x_1 + \varepsilon \left( \frac{x_2^3}{3} - x_2 \right)$$

③ Set  $V = \frac{1}{2} (x_1^2 + x_2^2)$ ,

$$\begin{aligned} \dot{V} &= \frac{dV}{dt} = x_1 \dot{x}_1 + x_2 \dot{x}_2 = x_1 x_2 - x_1 x_2 + \varepsilon \left( \frac{x_2^4}{3} - x_2^2 \right) \\ &= -\varepsilon x_2^2 \left( 1 - \frac{x_2^2}{3} \right) \end{aligned}$$

④ For  $x_2^2 < 3$  or  $x_2 < \pm\sqrt{3}$ ,  $\Rightarrow \dot{V} < 0$

$\Rightarrow$  With  $V = \frac{1}{2} (x_1^2 + x_2^2) > 0$  &  $\dot{V} < 0$ , the system is  
Lyapunov stable

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#2

## ② Application of the Lyapunov Theorems

\* Example #6 (Dynamic Stability) Turbulent flow in a channel

① Governing equation:

$$\frac{\partial u}{\partial t} = \frac{1}{R} \frac{\partial^2 u}{\partial x^2} + u - 2u \frac{\partial u}{\partial x} - R u \int_0^1 u^2 dx \longrightarrow \text{Nonlinear}$$

where  $u = u(x, t)$  = velocity field of a turbulent disturbance  
 $(u=0 \Rightarrow \text{turbulence vanishes})$

$$\textcircled{2} \quad \underline{\text{B.C.}}: \quad u(0, t) = u(1, t) = 0 \quad x \in [0, 1].$$

$$\textcircled{3} \quad T = \int_0^1 u^2 dx, \quad \frac{\partial u}{\partial t} = \frac{1}{R} \frac{\partial^2 u}{\partial x^2} + u \longrightarrow \text{Linear}$$

$$\begin{aligned} \frac{dT}{dt} &= \dot{T} = 2 \int_0^1 u \left( \frac{1}{R} \frac{\partial^2 u}{\partial x^2} + u \right) dx = 2 \int_0^1 \left[ u^2 - \frac{1}{R} \left( \frac{\partial u}{\partial x} \right)^2 \right] dx \\ &\quad \left( \because \int u \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) dx = u \frac{\partial u}{\partial x} - \int \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} dx \right) \end{aligned}$$

$$\Rightarrow \dot{T} \leq 2 \int_0^1 \left( 1 - \frac{\pi^2}{R} \right) u^2 dx = -2 \left( \frac{\pi^2}{R} - 1 \right) T$$

When  $R < \pi^2$ ,  $\dot{T} < 0 \Rightarrow$  Lyapunov stable (asymptotically stable).

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