Electromagnetics. Electrostatics.

Note Title 7/5/2016

Mathematical preliminaries:

(1) Consider the divergence theorem:

(1.1) $$\int \bar{\nabla}\cdot\bar{A}\, d^3r = \oint \bar{A}\cdot d\vec{a} = \oint \bar{A}\cdot\vec{n}\, da$$

with a special case when

$\vec{A} = \varphi \vec{\nabla} \psi$ where φ, ψ are arbitrary scalar functions

$$\vec{\nabla}\cdot(\varphi\vec{\nabla}\psi) = (\vec{\nabla}\varphi)\cdot(\bar{\nabla}\psi) + \varphi\nabla^2\psi$$

$$\vec{\nabla}\cdot(\psi\bar{\nabla}\varphi) = (\bar{\nabla}\varphi)\cdot(\bar{\nabla}\psi) + \psi\nabla^2\varphi$$

(1.2) $$\Rightarrow \boxed{\int_V(\varphi\nabla^2\psi - \psi\nabla^2\varphi)\,d^3r = \oint_S(\varphi\vec{\nabla}\psi - \psi\vec{\nabla}\varphi)\cdot\vec{n}\,da = \oint_S\left(\varphi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\varphi}{\partial n}\right)da}$$

and

(1.3) $$\oint_S \varphi\frac{\partial\psi}{\partial n}\,da = \int_V\left[(\bar{\nabla}\varphi)\cdot(\bar{\nabla}\psi) + \varphi\nabla^2\psi\right]d^3r$$

The latter equation can be used to prove the uniqueness of a solution for Poisson eq. within some volume

Indeed, suppose that Poisson eq has two solutions, φ_1, and φ_2.

(1.4) Then, $\boxed{u = \varphi_1 - \varphi_2}$ satisfies Laplace eq within the same volume

(1.2) with $\varphi = \psi = u \Rightarrow$

$$\int [\cancel{u\nabla^2 u} + (\vec{\nabla} u)\cdot(\vec{\nabla} u)]\, d^3r = \oint_S u \frac{\partial u}{\partial n}\, da$$

(1.5) $\Rightarrow$ $$\boxed{\int (\vec{\nabla} u)^2 d^3r = \oint_S u \frac{\partial u}{\partial n}\, da}$$

$\Rightarrow$ if either

(1.6) $$\begin{cases} u\big|_S \equiv 0 & (\text{Dirichlet' B.C}) \\ \text{or } \frac{\partial u}{\partial n}\Big|_S \equiv 0 & (\text{Neuman B.C}) \end{cases}$$

$\Rightarrow$ $(\nabla u)^2 \equiv 0$ within the volume

$\Leftrightarrow$ at best $\varphi_1 - \varphi_2 = $ const

Note: 1) mixed B.C [part Dirichlet, part Neuman does not work];

2) Finite volume assumed (∞ requires special treatm)

2 Dirac δ-functions

δ-function is an object that satisfies:

(1.7) $\rightarrow$ $$\boxed{\int \delta(\bar{r}-\bar{r}_0)\, f(\bar{r})\, d^n\bar{r} = f(\bar{r}_0)}$$

$\rightarrow$ $\delta(\bar{r}) = 0, \quad \vec{r} \neq 0$

$\rightarrow$ $\int \delta(\bar{r})\, d^n r = 1$

$\rightarrow$ δ-function can be considered as a

$\lim_{n\to\infty} \delta_n(\bar{r})$, ($\delta$-sequence)

$\rightarrow$ All properties of δ-function are related back to (1.1)

For example; consider $\delta(ax)$ in 1D:

$$\int \delta[a(x-x_0)] f(x)\, dx = \frac{1}{|a|} \int \delta(\xi-\xi_0) f\left(\frac{\xi}{a}\right) d\xi = \frac{f(\xi_0/a)}{|a|} =$$

$$\Rightarrow \delta(a[x-x_0]) = \frac{\delta(x-x_0)}{a}$$

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(3) δ-function is often used in calculation of Green's functions. For example, for Poisson's eq:

(1.8)
$$\Delta G(\bar{r}, \bar{r}_0) = -4\pi\, \delta(r-r_0)$$

Then solution to general Poisson's eq:

(1.9)
$$\Delta V = -4\pi f(r)$$

is derived as a linear combination:

(1.10)
$$V(r) = \int \sigma(r_0)\, G(r, r_0)\, dr_0 \,,$$

From (1.2 ... 1.4), obtain:

operates on r, not r_0

$$\Delta V = \Delta \int \sigma(r_0)\, G(r, r_0)\, dr_0 = -4\pi f(r)$$

$$\int \sigma(r_0) \left[-4\pi\, \delta(\bar{r}-\bar{r}_0)\right] dr_0 = -4\pi f(\bar{r})$$

$$-4\pi\sigma(r) = -4\pi f(r)$$

(1.11)
$$\Rightarrow \boxed{V(r) = \int G(\bar{r}, \bar{r}_0)\, f(r_0)\, dr_0}$$

But what about boundary conditions?

Here we note that $\tilde{G}(r, r_0)$ is not unique.

thus,

(1.12) $$\begin{cases} G(r, r_0) = \tilde{G}(r, r_0) + F(r, r_0), \text{ where} \\ \Delta_r F(r, r_0) = 0 \end{cases}$$

Therefore, we have to go back to Eq. (1.2) with

$\varphi = V$; $\psi = G(r, r_0)$

$$\oint_S \left[V \frac{\partial G}{\partial n} - G(r, r_0) \frac{\partial V}{\partial n}\right] d^2a =$$

$$\int_V \left[V(r)\, \nabla^2 G(r, r_0) - G(r, r_0) \nabla^2 V\right] d^3r =$$

$$= \int_V \left[V(r)\left(-4\pi\, \delta(r - r_0)\right) + G(r, r_0)\, 4\pi f(r)\right] d^3r$$

(1.13) $$\Rightarrow \boxed{V(r_0) = \int_V G(r, r_0) f(r)\, d^3r + \frac{1}{4\pi} \oint_S \left[G(r, r_0) \frac{\partial V(r)}{\partial n} - V(r) \frac{\partial G}{\partial n}\right] d^2a}$$

Note that it is impossible to arbitrarily define

$V\big|_S$ <u>and</u> $\frac{\partial V}{\partial n}\Big|_S$. therefore,

① Choose $G(r, r_0)\Big|_{r=S} = 0 \Rightarrow$

Solution to Dirichlet's problem

(1.14a) $$\boxed{V(r) = \int_V G(r, r_0) f(r_0)\, d^3r_0 - \frac{1}{4\pi} \int V(r_0) \frac{\partial G(r, r_0)}{\partial n_0}\, d^2r_0}$$

② Choose $\frac{\partial G}{\partial n}(r, r_0)\Big|_{r=S} = -\frac{4\pi}{S}$ $\left[\int \bar{\nabla} \cdot \bar{\nabla} G\, dV = \oint \frac{\partial G}{\partial n} d^2a = -4\pi\right]$

Solution to Neumann problem

(often, outside S, part of $S = \infty$, $\Rightarrow \langle V \rangle = 0$)

(1.14b) $$\boxed{V(r) = \int_V G(r, r_0) f(r_0)\, d^3r_0 + \frac{1}{4\pi} \oint G(r, r_0) \frac{\partial V(r_0)}{\partial n}\, d^2r_0 - \langle V \rangle\Big|_S}$$

Starting from Coulomb's law,

(1.15) $$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \hat{r} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^3} \vec{r}$$

we can introduce Electric field as:

(1.16) $$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^3} \vec{r}}, \quad \vec{F} = Q \cdot \vec{E}(r)$$

$$\oint \vec{E} \cdot d\vec{a} = \frac{1}{4\pi\epsilon_0} \oint \frac{q}{r^2} \underbrace{\hat{r} \cdot d\vec{a}}_{d\Omega} = \frac{1}{\epsilon_0} q_{enc}$$

(1.16b) or, equivalently, $$\boxed{\oint \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} \int \rho(r) d^3 r}$$

Thus, electric field is a force per unit charge.

Experimentally, $\vec{E}$ ($\vec{F}$) were found to satisfy:

→ $1/r^2$ dependence

→ superposition principle

→ central force nature of $\vec{F}$, $\vec{E}$

Using divergence theorem,

$$\oint \vec{E} \cdot d\vec{a} = \int \mathrm{div}\, \vec{E}\, d^3 r = \frac{1}{\epsilon_0} \int \rho(r) d^3 r$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}}$$

(1.16) can be generalized to:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r_0)(\vec{r} - \vec{r}_0)}{|r - r_0|^3} d^3 r_0$$

$$\bar{\nabla}\times\bar{E} = \frac{1}{4\pi\epsilon_0}\bar{\nabla}\times\int\frac{\rho(r_0)\,(\bar{r}-\bar{r}_0)}{|\bar{r}-\bar{r}_0|^3}\,d^3r_0 =$$

$$= \frac{1}{4\pi\epsilon_0}\int\rho(r_0)\,d^3r_0\cdot\bar{\nabla}_r\times\frac{(\bar{r}-\bar{r}_0)}{|\bar{r}-\bar{r}_0|^3} = 0$$

in Electrostatics.

(1.17) $$\boxed{\bar{\nabla}\times\bar{E} = 0} \iff$$

(1.18) $$\boxed{\vec{E} = -\vec{\nabla}\Phi}$$

Poisson eq

where Φ is scalar potential. Note:

(1.19) $$\nabla^2\Phi = \bar{\nabla}\cdot\bar{\nabla}\Phi = -\bar{\nabla}\cdot\vec{E} = -\frac{\rho}{\epsilon_0};\quad \boxed{\Delta\Phi = -\frac{\rho}{\epsilon_0}}$$

in Free-space: $E = \frac{1}{4\pi\epsilon_0}\frac{q\hat{r}}{r^2} \Rightarrow$

(1.20) $$\boxed{\Phi = \frac{1}{4\pi\epsilon_0}\frac{q}{|\bar{r}-\bar{r}_0|}}$$

in case of arbitrary charge distribution, need Green function; try:

(1.21) $$\boxed{G(r,r_0) = \frac{1}{|\bar{r}-\bar{r}_0|}} = \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}$$

$$\nabla G = -\frac{\bar{r}-\bar{r}_0}{|\bar{r}-\bar{r}_0|^3}$$

$$\Delta G = -\frac{\vec{\nabla}\cdot(\bar{r}-\bar{r}_0)}{|\bar{r}-\bar{r}_0|^3} - (\bar{r}-\bar{r}_0)\cdot\bar{\nabla}\left(\frac{1}{|\bar{r}-\bar{r}_0|^3}\right) = -\frac{3}{|\bar{r}-\bar{r}_0|^3}$$

$$-(\bar{r}-\bar{r}_0)\cdot\left[-3\frac{(\bar{r}-\bar{r}_0)}{|\bar{r}-\bar{r}_0|^5}\right] = 0,\quad \forall\,\bar{r}\neq\bar{r}_0$$

$$\int\Delta G\,d^3v = \oint\bar{\nabla}G\cdot d^2\bar{a} = -\oint\frac{(\bar{r}-\bar{r}_0)}{|\bar{r}-\bar{r}_0|^3}\cdot d^2\bar{a} = -4\pi$$

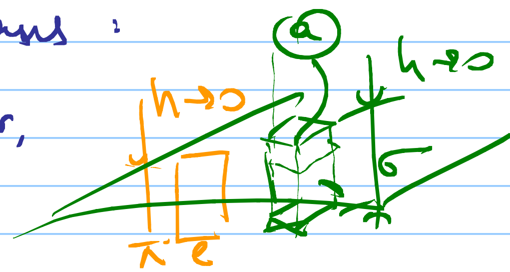
[sphere]

$\Rightarrow\ \Delta G = -4\pi\,\delta(r-r_0) \Rightarrow \frac{1}{|\bar{r}-\bar{r}_0|}$ is a proper Green function

⇒ for arbitrary charge distribution:

(1.22) $$\boxed{\Phi(r) = \int \frac{\rho(r_0)}{4\pi\epsilon_0} G(r, r_0)\, d^3r_0 = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r_0)}{|r - r_0|} d^3r_0}$$

On boundary conditions:



Consider a charged conductor, with surface charge σ

Then, $\int_a \bar{\nabla} \times \bar{E}\, da = \oint_l \bar{E} \cdot d\bar{l} =$

(1.23)
$$= \Delta \bar{E}_\tau \cdot l = 0 \Rightarrow \boxed{E_\tau = \text{const}}$$

$$\int \bar{\nabla} \cdot \bar{E}\, d^3v = \oint \bar{E} \cdot d\vec{a} = \frac{\int \rho\, d^3v}{\epsilon_0}$$

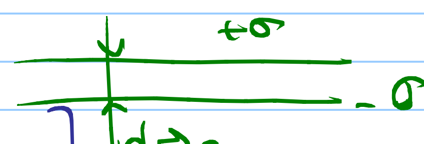
$$\Delta E_n \cdot a = \frac{\sigma \cdot a}{\epsilon_0} \Rightarrow \boxed{\Delta E_n = \frac{\sigma}{\epsilon_0}}$$

Note, $\Phi(r)$ remains continuous (even within the conductor) and can be calculated using (1.22):

$$\Phi(r) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r_0)\, dh\, d^2a_0}{|r - r_0|} = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(r_0)\, d^2a_0}{|r - r_0|}$$

d^3r_0

Consider now a bi-layer shown at right.



Now, $$\Phi(r) = \frac{1}{4\pi\epsilon_0} \int \sigma(r_0)\, d^2a_0 \left[\frac{1}{|r - r_0 + d|} - \frac{1}{|r - r_0|} \right]$$

$$= \frac{1}{4\pi\epsilon_0} \int \sigma(r_0)\, d^2a_0 \left[\vec{d} \cdot \vec{\nabla}_0 \frac{1}{|r - r_0|} \right] = \frac{1}{4\pi\epsilon_0} \int \frac{\vec{D} \cdot (\bar{r} - \bar{r}_0)}{|r - r_0|^3} d^2a_0$$

$\vec{D}$

Note that due to $d \to 0$ transition Φ becomes discontinuous, with

(1.23a) $$\Delta \Phi = D/\epsilon_0$$

Note that potential has a clear physical meaning:

$\vec{F} = Q\vec{E}$, the work done against this force:

$$W = -\int_a^b \vec{F}\cdot d\vec{l} = Q\int \vec{\nabla}\varphi \cdot d\vec{l} = Q\int_a^b d\phi \Rightarrow$$

(1.24) $$\boxed{W = \varphi(b) - \varphi(a)}$$

(1.25) In particular: $W_0 = -\oint \vec{F}\cdot d\vec{l} = -Q\oint \vec{E}\cdot d\vec{l} =$

$$= -Q\int \vec{\nabla}\times\vec{E}\cdot d\vec{a} = 0$$

The energy of a system of charges:

$$W = \sum_{i,j<i} Q_i\varphi_j(x_i) = \sum_{i,j<i} \frac{Q_iQ_j}{4\pi\epsilon_0}\frac{1}{|\bar{r}_i - \bar{r}_j|} =$$

$$= \frac{1}{8\pi\epsilon_0}\sum_{i,j\neq i} \frac{Q_iQ_j}{|\bar{r}_i - \bar{r}_j|}$$

linear comb.

Note that $\varphi_j = \sum_{i\neq j} \frac{Q_i}{4\pi\epsilon_0|\bar{r}_i - \bar{r}_j|} = \sum_i p_{ij}Q_i$

(1.26) $\Rightarrow$ $$\boxed{Q_i = \sum_j C_{ij}\varphi_j}$$

mutual capacitances $(i=j)$

— " — inductances $(i\neq j)$

$$\Rightarrow W = \sum_{i,j} C_{ij}\varphi_i\varphi_j$$

Note that the change in energy

$$\delta U = \sum_i \varphi_i\,\delta Q_i = \sum_j Q_j\,\delta\varphi_j$$

$$\rightarrow \frac{\partial U}{\partial Q_i} = \varphi_i \;;\; \frac{\partial U}{\partial \varphi_j} = Q_j$$

(1.27) $$\frac{\partial^2 U}{\partial Q_i\partial\varphi_j} = \frac{\partial\varphi_i}{\partial Q_j} = C_{ij} = \frac{\partial^2 U}{\partial Q_j\partial\varphi_i} = \frac{\partial\varphi_j}{\partial Q_i} = C_{ji}$$

Note that since the energy is >0,

(1.27a) $\boxed{C_{ii} > 0}$;
At the same time, $\boxed{C_{ij} < 0}$, $i \neq j$

[to prove this, consider a system of conductors with $\varphi_1 > 0$, all $\varphi_i = 0$, $i > 1$ (grounded). then

$$Q_i = C_{1i}\,\varphi_1 < 0$$ [since induced charge is opposite to original charge)

$\nabla^2 \varphi = -\frac{\rho}{\epsilon_0}$

In continuous limit:

$$W = \frac{1}{8\pi\epsilon_0} \iint \frac{\rho(\bar{r})\,\rho(\bar{r}_0)}{|\bar{r} - \bar{r}_0|} d^3r\, d^3r_0 = \frac{1}{2}\int \rho(r)\,\varphi(r)\,d^3r$$

$$= -\frac{1}{2}\int (\nabla^2 \varphi)\,\varphi(r)\,d^3r = \frac{1}{2}\int (\vec{\nabla}\varphi)^2 d^3r \Rightarrow$$

(1.28) $$\boxed{W = \frac{1}{2}\int \bar{E}^2 d^3r} > 0$$

Let's take one step back and show that the requirement for a physical system to have minimum energy of the form (1.28) yields Poisson eq.

(1.29) $$\boxed{W = \frac{1}{2}\int (\vec{\nabla}\varphi)\cdot(\vec{\nabla}\varphi)\,d^3r - \int g\,\Phi\,d^3r}$$

(1.30) $$\delta W = \int (\vec{\nabla}\varphi)\cdot(\vec{\nabla}\,\delta\varphi)\,d^3r - \int g\,\delta\Phi\,d^3r \overset{\text{(Green identity)}}{=}$$

$$= -\int (\nabla^2\Phi + g)\,\delta\varphi\,d^3r + \oint_S \delta\varphi \frac{\partial\varphi}{\partial n}\,da = 0$$

$\delta\varphi\big|_S = 0$

(1.30a) $\Rightarrow$ $$\boxed{\nabla^2 \Phi = -g}$$

Finally, consider numerical solutions to Laplace eq. Assume that the potential $\varphi(x,y)$ is represented on a square grid:

(1.31) $$\begin{cases} x_i = x_0 + i\cdot h; \quad y_j = y_0 + j\cdot h, \quad h \ll 1. \\ \varphi_{ij} = \varphi(x_i, y_j) \end{cases}$$

(1.32) $$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial \varphi}{\partial x}\right] \simeq \frac{1}{h}\left[\frac{\varphi_{i+1,j} - \varphi_{i,j}}{h} - \frac{\varphi_{ij} - \varphi_{i-1,j}}{h}\right] =$$

$$= \frac{1}{h^2}\left[\varphi_{i+1,j} + \varphi_{i-1,j} - 2\varphi_{ij}\right]$$

$\Rightarrow \Delta\varphi = 0 \Leftrightarrow$

$$\frac{1}{h^2}\left[\varphi_{i+1,j} + \varphi_{i-1,j} + \varphi_{i,j+1} + \varphi_{i,j-1} - 4\varphi_{ij}\right] = 0$$

(1.33) $$\Rightarrow \boxed{\varphi_{i,j} = \frac{1}{4}\left[\varphi_{i+1,j} + \varphi_{i-1,j} + \varphi_{i,j+1} + \varphi_{i,j-1}\right]}$$

Relaxation technique

1. Set B.C for $\varphi|_S$
2. calculate internal φ_{ij} using (1.33)
3. check convergence

not converged

OK