Laplace equation in cylindrical coordinates

Tuesday, August 23, 2016 10:05 AM

Laplace equation in cylindrical coordinates:

(3.1) $$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial\varphi^2}+\frac{\partial^2 V}{\partial z^2}=0$$

Use separation of variables:

(3.2) $$V=R(\rho)\cdot\Phi(\varphi)\cdot Z(z)$$

$$\Rightarrow \frac{\Phi\cdot Z}{\rho}R'+\Phi\cdot Z\cdot R''+\frac{R\cdot Z\Phi''}{\rho^2}+\Phi\cdot R Z''=0$$

$$\frac{R''}{R}+\frac{R'}{\rho R}+\frac{1}{\rho^2}\underbrace{\frac{\Phi''}{\Phi}}_{-m^2}+\underbrace{\frac{Z''}{Z}}_{k^2}=0$$

(3.3)
$$\Rightarrow \Phi''=-m^2\Phi \Rightarrow \Phi_m\propto e^{\pm im\varphi} \text{ or } \begin{matrix}\cos\\ \sin\end{matrix}(m\varphi)$$

$$\Rightarrow Z''=k^2 Z \Rightarrow Z_k\propto e^{\pm kz} \text{ or } \begin{matrix}\text{ch}\\ \text{sh}\end{matrix}(kz)$$

$$\Rightarrow \boxed{R''+\frac{R'}{\rho}+\left(k^2-\frac{m^2}{\rho^2}\right)R=0}$$

introducing $x=k\rho$, obtain:

$$\frac{d^2R}{dx^2}+\frac{1}{x}\frac{dR}{dx}+\left(1-\frac{m^2}{x^2}\right)R=0$$

$$\Rightarrow R_{mk}=J_m(k\rho)\quad\left(Y_m(k\rho),\ H_m^{\pm}(k\rho)\text{ also possible}\right)$$

where $Y_m \equiv N_m(x) = \dfrac{J_m(x)\cos m\pi - J_{-m}(x)}{\sin m\pi}$

$$H_m^{\pm} \equiv H_m^{(1,2)}(x) = J_m(x) \pm i\,Y_m(x)$$

Properties of Bessel functions

Generating function

(3.4) $$\boxed{g(x,t) = e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = \sum_n J_n(x)\,t^n}$$

$$\frac{\partial g}{\partial x} = \frac{1}{2}\left(t - \frac{1}{t}\right) g(x,t) = \sum_n \frac{J_n}{2}\,t^{n+1} - \sum_n J_n \frac{t^{n-1}}{2} = \sum_n J_n'\,t^n$$

(3.4a) @ $t^n \Rightarrow$ $$\boxed{\frac{1}{2}\left(J_{n-1}(x) - J_{n+1}(x)\right) = J_n'(x)}$$

$$\frac{\partial g}{\partial t} = \frac{x}{2}\left(1 + \frac{1}{t^2}\right) g(x,t) = \sum_n \frac{x}{2}\left(J_n\,t^n + J_n\,t^{n-2}\right) = \sum_n n J_n\,t^{n-1}$$

(3.4b) @ $t^{n-1} \rightarrow$ $$\boxed{\frac{x}{2}\left(J_{n-1}(x) + J_{n+1}(x)\right) = n J_n(x)}$$

Note: it is possible to start with recurrence relation & derive Bessel eq. (homework)

Eq. (33) can be written in the self-adjoint form:

$$\frac{d}{d\rho}\left[\rho \frac{d J_m(k\rho)}{d\rho}\right] + \frac{k^2\rho^2 - m^2}{\rho} J_m(k\rho) = 0$$

weighting function

$$\Leftrightarrow \quad Lu = \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) + qu = \lambda w(x) u$$

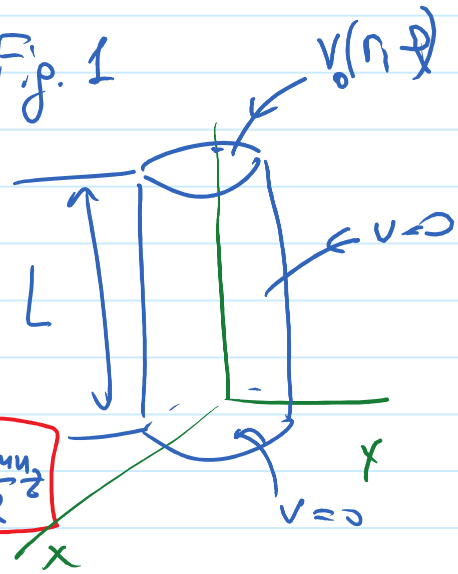
Since eigenfunctions of self-adjoint operators are orthogonal,

(3.5) $$\boxed{\int_0^a \rho\, J_m\left(\frac{\alpha_{mn}\rho}{a}\right) J_m\left(\frac{\alpha_{m\ell}\rho}{a}\right) d\rho = \delta_{n\ell}\, \frac{a^2}{2}\left[J_{m+1}(\alpha_{m\ell})\right]^2}$$

where $\alpha_{m\ell}$ is ℓ-th zero of J_m

Boundary-value problems in Cylindrical cord. (example)

Consider a problem shown in Fig. 1



Then $V=\sum_{m,k}\left(a_m\cos m\varphi + b_m\sin m\varphi\right)J_m(k\rho)\sinh kz$

B.C. $J_m(kR)=0 \Rightarrow kR=\alpha_{mn}$

(3.) $$\boxed{\Rightarrow V=\sum_{m,n}\left(a_{mn}\cos m\varphi + b_{mn}\sin m\varphi\right)J_m\left(\frac{\alpha_{mn}}{R}\rho\right)\sinh\frac{\alpha_{mn}}{R}z}$$

$$V(z=L)=\sum_{m,n}\left(a_{mn}\cos m\varphi + b_{mn}\sin m\varphi\right)J_m\left(\frac{\alpha_{mn}}{R}\rho\right)\sinh\frac{\alpha_{mn}L}{R}=V_0(\rho,\varphi)$$

$$*\int_0 J_m\left(\alpha_{ml}\frac{\rho}{R}\right)\rho \Rightarrow$$

$$\sum_m\left(a_{ml}\cos m\varphi + b_{ml}\sin m\varphi\right)\frac{R^2}{2}J^2_{m+1}(\alpha_{ml})\sinh\frac{\alpha_{ml}L}{R}=\tilde V_{ml}(\varphi)$$

where $\tilde V_{ml}=\int_0^R V(\rho,\varphi)\,\rho\, J_m\left(\alpha_{ml}\frac{\rho}{R}\right)d\rho$

(3.) $$a_{ml}\frac{R^2}{2}J^2_{m+1}(\alpha_{ml})\sinh\frac{\alpha_{ml}L}{R}=V^c_{ml}$$

$$a_{ml} \; 2 \; J_{m+1}(\alpha_{ml}) \qquad R = V_{ml}$$

(3.7)

$$b_{ml} \;-\; \text{"} \;\; \underline{\qquad\qquad} \;=\; V^{s}_{ml}$$

where $$V^{c,s}_{ml} = \frac{1}{\pi} \int_0^{2\pi} d\varphi \, \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(m\varphi) \int_0^R V(\rho,\varphi)\,\rho\, J_m\!\left(\alpha_{ml}\frac{\rho}{R}\right) d\rho$$

Laplace equation in spherical coordinates

Wednesday, August 24, 2016 12:32 AM

Laplace equation in spherical coordinates:

$$\boxed{\frac{1}{r}\frac{\partial^2}{\partial r^2}(rV)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}=0} \quad (3.8)$$

Once again, use separation of variables:

$$V=R(r)\cdot\Phi(\varphi)\cdot\Theta(\theta)$$

$$\frac{\Phi\Theta}{r}\frac{d^2}{dr^2}(rR)+\frac{R\Phi}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{R\Theta}{r^2\sin^2\theta}\frac{d^2\Phi}{d\varphi^2}=0 \quad (3.8a)$$

$$\Rightarrow \Phi_m''=-m^2\Phi_m$$

(3.8a) now becomes:

$$\frac{1}{rR}\frac{d^2}{dr^2}(rR)+\frac{1}{r^2}\underbrace{\left[\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\frac{m^2}{\sin^2\theta}\right]}_{-\ell(\ell+1)}=0 \quad (3.8b)$$

$\Rightarrow$ Angular Part

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]\Theta_{m\ell}=0 \quad (3.9)$$

Introducing $\xi=\cos\theta$

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{d\Theta}{d\xi}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-\xi^2}\right]\Theta=0 \quad (3.9a)$$

$$\Theta_{m\ell}(x) = P_\ell^m(x) \text{ or } Q_\ell^m(x) \leftarrow \text{associated Legendre functions}$$

$$\Theta_{m\ell} = P_\ell^m(\cos\theta)$$

Angular part: spherical harmonics:

(3.9) $$Y_{\ell m}(\theta, \varphi) \propto P_\ell^m(\cos\theta)\, e^{im\varphi}$$

More information on Legendre functions

Start with $m = 0$ (realized in azimuthically-symmetric case)

Eq. (3.9) becomes:

(3.10) $$\boxed{\frac{d}{dx}\left[(1-x^2)\frac{dP_\ell(x)}{dx}\right] + \ell(\ell+1)P_\ell(x) = 0} \leftarrow \text{Legendre eqn}$$

Once again, we have self-adjoined operator

(3.11) $$\Rightarrow \boxed{\int_{-1}^{1} P_\ell(x)\, P_m(x)\, dx = \frac{2\delta_{m\ell}}{2\ell+1}}$$

Solutions to Legendre eqn can be represented as series:

$$P_\ell(x) = \sum_n a_n x^{n+k}$$

$$P_\ell'(x) = \sum_n (n+k)\, a_n x^{n+k-1}$$

$$P_\ell''(x) = \sum_n (n+k)(n+k-1)\, a_n x^{n+k-2}$$

$$(3.10) \Rightarrow P_\ell'' - x^2 P_\ell'' - 2x P_\ell' + \ell(\ell+1) P_\ell = 0$$

(3.12)
$$\Rightarrow \begin{cases} \sum_n (n+k)(n+k-1) a_n x^{n+k-2} + \\ \sum \Big[\underbrace{-(n+k)(n+k-1) - 2(n+k)}_{-(n+k)(n+k+1)} + \ell(\ell+1) \Big] a_n x^{n+k} = 0 \end{cases}$$

for series solution to exist, for $n=0$:

(3.13)

$$k(k-1) = 0 \Rightarrow \boxed{k=0} \text{ or } \boxed{k=1}$$

$$\xrightarrow{\text{indicial equ}}$$

for every other n: (looking for $\propto x^{n+k}$):

$$(n+k+2)(n+k+1) a_{n+2} + \left[\ell(\ell+1) - (n+k)(n+k+1)\right] a_n = 0$$

$$\Rightarrow \boxed{a_{n+2} = \frac{(n+k)(n+k+1) - \ell(\ell+1)}{(n+k+2)(n+k+1)} a_n} \leftarrow \text{recurrence relation}$$

$n = 0, 2, 4, \ldots$

$k=0$: series terminates for even ℓ @ $n=\ell$ $[P_\ell(x)]$
infinite series for odd ℓ $[Q_\ell(x)]$

$k=1$: series terminates for odd ℓ @ $n=\ell-1$ $[P_\ell(x)]$
infinite series for even ℓ $[Q_\ell(x)]$

$\Rightarrow P_\ell(x)$ is a polynomial of order ℓ

(3.14) Rodrigues formula: $$\boxed{P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell}$$

Properties of Legendre polynomials can be derived from generating function:

generating function:

(3.15) $$g(x,t) = \boxed{\frac{1}{\sqrt{1-2xt+t^2}} = \sum_n P_n(x)\, t^n}$$

$$\frac{\partial g}{\partial x} = \frac{t}{(1-2xt+t^2)^{3/2}} = \frac{t\, g(x,t)}{1-2xt+t^2} \Rightarrow$$

$$\sum P_n(x)\, t^{n+1} = (1-2xt+t^2) \sum_n P_n'(x)\, t^n$$

$$@\, t^{n+1} \quad P_n(x) = P_{n+1}' - 2xP_n' + P_{n-1}'$$

(3.16a) $$\boxed{P_{n+1}'(x) + P_{n-1}'(x) = P_n(x) + 2xP_n'(x)}$$

$$\frac{\partial g}{\partial t} = \frac{(x-t)\, g(x,t)}{1-2xt+t^2} \Rightarrow$$

$$\sum_n x P_n(x) t^n - \sum_n P_n(x) t^{n+1} =$$

$$\sum n P_n(x) t^{n-1} - 2x \sum_n n P_n t^n + \sum n P_n(x) t^{n+1}$$

$$@\, t^n: \; xP_n - P_{n-1} = (n+1)P_{n+1} - 2xnP_n + (n-1)P_{n-1}$$

(3.16b) $$\boxed{(n+1)P_{n+1}(x) + nP_{n-1}(x) = (2n+1)\, x\, P_n(x)}$$

Associated Legendre Functions:

(3.17) $$\begin{cases} P_\ell^m(x) = (-1)^m (1-x^2)^{m/2} \dfrac{d^m}{dx^m} P_\ell(x) \\[2ex] P_\ell^m(x) = (-1)^m (1-x^2)^{m/2} \dfrac{d^{\ell+m}}{dx^{\ell+m}} (x^2-1)^\ell \end{cases}$$

Note that Eq. (3.9a) represents self-adjoint operator with respect to ℓ $\Rightarrow$ P_ℓ^m are orthogonal:

(3.18) $$\boxed{\int_{-1}^{1} P_\ell^m(x)\, P_n^m(x)\, dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\,\delta_{\ell n}}$$

Spherical harmonics:

(3.19) $$\boxed{Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\varphi}}$$

(3.20) Note that $$\boxed{\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi) = \delta(\varphi-\varphi')\,\delta(\cos\theta-\cos\theta')}$$

Radial part of Laplace eqn:

(3.21) $$\frac{1}{r}\frac{d^2}{dr^2}(rR) - \frac{\ell(\ell+1)}{r^2}R = 0$$

$$R_\ell = r^\alpha ;$$

$$r(\alpha+1)\alpha r^{\alpha-1} - \ell(\ell+1) r^\alpha = 0 \Rightarrow$$

$$\alpha(\alpha+1) - \ell(\ell+1) = 0$$

$$\alpha^2 + \alpha - \ell(\ell+1) = 0$$

(3.22) $$\alpha = \frac{-1 \pm \sqrt{1+4\ell^2+4\ell}}{2} = \frac{-1\pm(2\ell+1)}{2} = \begin{cases}\ell \\ -\ell-1\end{cases}$$

Boundary value problems in spherical coordinates

Wednesday, August 24, 2016 12:28 AM

Boundary - value problems in spherical coord.

General solution:

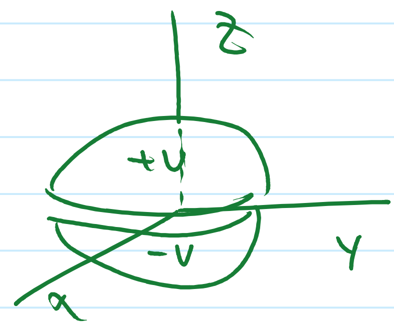
(3.23) $$V(r,\theta,\varphi) = \sum_{\ell,m}\left[\alpha_{\ell m} r^\ell + \beta_{\ell m}\frac{1}{r^{\ell+1}}\right] Y_{\ell m}(\theta,\varphi)$$

where coefs $\vec{\alpha}, \vec{\beta}$ need to be determined

Axially - symmetric problems:

(3.23a) $$V(r,\theta) = \sum_\ell \left[a_\ell r^\ell + \frac{b_\ell}{r^{\ell+1}}\right] P_\ell(\cos\theta)$$

Example: find potential inside/outside the sphere where potential is fixed @ $\pm V$



The potential inside the sphere:

(3.24)
$$V(r,\theta) = \sum_\ell a_\ell r^\ell P_\ell(\cos\theta)$$

The potential outside the spheres

$$V(r,\theta) = \sum \frac{b_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

In both cases:

$$V(R,\theta) = \sum \alpha_\ell P_\ell(\cos\theta) = \pm V_0$$

where $\alpha_\ell = a_\ell R^\ell$ or $\frac{b_\ell}{\ }$

where $\alpha_\ell = a_\ell R^\ell$ or $\dfrac{b_\ell}{R^{\ell+1}}$

To find the coefficients α_ℓ:

$$\sum \alpha_\ell P_\ell(\cos\theta) = \pm V_0 \qquad \left\{ \times \int P_n(\cos\theta)\, d(\cos\theta) \right.$$

$$\alpha_n \frac{2}{2n+1} = \int_{-1}^{1} \pm V_0 P_n(x)\, dx = -\int_{-1}^{0} V_0 P_n(x)\, dx + \int_0^1 V_0 P_n(x)\, dx = \mp \int_0^1 V_0 P_n(-x)\, dx + \int_0^1 V_0 P_n(x)\, dx$$

$$= \begin{cases} n - \text{odd} \Rightarrow 2\int_0^1 V_0 P_n(x)\, dx \\ n = \text{even} \Rightarrow 0 \end{cases}$$

(3.25) for odd n: $\boxed{\alpha_n = (2n+1) V_0 \int_0^1 P_n(x)\, dx}$

Field in a conical hole:

Angular part of Laplace eqn

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_\nu(x)}{dx}\right] + \nu(\nu+1) P_\nu(x) = 0$$

$$V(r,\theta) = \sum_\nu \alpha_\nu r^\nu P_\nu(\cos\theta)$$

(3.26) where ν are chosen such as $\boxed{P_\nu(\cos\theta_0) = 0}$

for small radii:

for small radii

$$
(3.27)\quad \begin{cases} V(r,\theta) \simeq \alpha_\nu \, r^\nu \, P_\nu(\cos\theta) & \text{where } \nu \text{ is the smallest root of eqn. above} \\ E_r = -\dfrac{\partial V}{\partial r} \,; \quad E_\varphi = -\dfrac{1}{r}\dfrac{\partial V}{\partial \varphi} \,; \quad \sigma(r) = -\dfrac{1}{4\pi} E_\theta \Big|_{\theta_0} \end{cases}
$$

Green function expansion

Wednesday, August 24, 2016 12:25 AM

Green function expansion:

$$(3.28) \quad \boxed{\frac{1}{|\vec{r}-\vec{r}'|} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y_{\ell m}^*(\theta',\varphi') Y_{\ell m}(\theta,\varphi)}$$

where $r_<, r_>$ are smaller/larger of r, r'

generalization to "outside" problem is also straightforward