Magnetostatics; Vector potential

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In contrast to electric fields, there is no magnetic charges. Magnetic field is generated with currents, related to moving charges via

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$

Magnetic induction $\vec{B} = \frac{\mu_0}{4\pi}\int \vec{j}(\vec{r}\,')\times\frac{\vec{r}-\vec{r}\,'}{|r-r'|^3}\,d^3r' \Rightarrow$

$$\vec{B} = \frac{\mu_0}{4\pi}\,\vec{\nabla}_r\times\int\frac{\vec{j}(\vec{r}\,')}{|r-r'|}\,d^3r'$$

Note that $\vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\,\vec{\nabla}\cdot\left[\vec{\nabla}\times\int\frac{\vec{j}(\vec{r}\,')}{|r-r'|}\,d^3r'\right] = 0$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\,\vec{\nabla}\times\vec{\nabla}\times\int\frac{\vec{j}(r')}{|r-r'|}\,d^3\bar{r} = \frac{\mu_0}{4\pi}\,\vec{\nabla}\left(\vec{\nabla}\cdot\int\frac{\vec{j}(r')\,d^3r}{|r-r'|}\right) -$$

$$-\frac{\mu_0}{4\pi}\,\nabla^2\int\frac{\vec{j}(r')\,d^3r}{|r-r'|} = \quad \left[\vec{\nabla}\times\vec{\nabla}\times = \vec{\nabla}(\vec{\nabla}\cdot\ldots) - \Delta\ldots\right]$$

$$-\frac{\mu_0}{4\pi}\,\vec{\nabla}\int\vec{j}\cdot\vec{\nabla}_r\left(\frac{1}{|r-r'|}\right)d^3r' - \frac{\mu_0}{4\pi}\int j(r')\,\underbrace{\Delta\frac{1}{|r-r'|}}_{-4\pi\,\delta(r-r')}\,d^3r'$$

$$= -\mu_0\, j(r) - \frac{\mu_0}{4\pi}\,\bar{\nabla}\int\bar{j}(r')\cdot\bar{\nabla}_{r'}\frac{d^3r'}{|r-r'|} =$$

$$= \mu_0\, j(r) + \frac{\mu_0}{4\pi}\,\bar{\nabla}\int\frac{\vec{\nabla}\cdot\vec{j}}{|r-r'|}\,d^3r'$$

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$$\Rightarrow \quad \vec{\nabla}\times\vec{B} = \mu_0 \vec{j}$$

$$\int \vec{\nabla}\times\vec{B}\, d^2a = \oint \vec{B}\cdot d\vec{l} = \mu_0 I$$

Since $\vec{\nabla}\cdot\vec{B} = 0$, we can introduce vector potential in the form

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

Note that if $\vec{A} \to \vec{A} + \vec{\nabla}\psi$ (Gauge transformation)

$$\vec{B} \to \vec{B} + \vec{\nabla}\times(\vec{\nabla}\psi) = \vec{B}$$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \mu_0 \vec{j}$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \Delta\vec{A} = \mu_0 \vec{j}$$

Coulomb gauge: $\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow \Delta\vec{A} = -\mu_0 \vec{j}$

(Poisson's eq). $\Rightarrow \vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{j}(r')}{|\vec{r}-\vec{r}'|}\, d^3r$

Note that when $\vec{j} = 0$,

$$\vec{\nabla}\times\vec{B} = 0$$

$\Rightarrow$ can introduce "magnetic scalar potential"

$$\vec{B} = -\vec{\nabla}\varphi_m,$$

reducing magnetic problems to electrostatics.

Magnetic field of a localized current distribution

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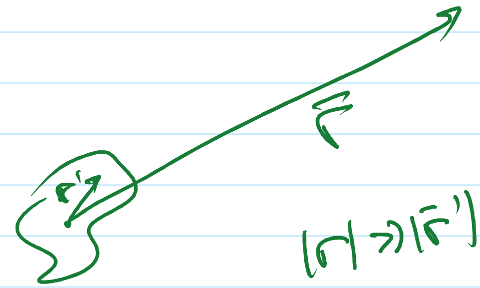
Consider localized current distribution:



We are interested in the fields far away from such currents.

$$\vec A = \frac{\mu_0}{4\pi}\int \frac{\vec j(r')}{|\vec r-\vec r'|}\, d^3r'$$

$$\frac{1}{|r-r'|} \simeq \frac{1}{|\vec r|} - \frac{\vec r-\vec r'}{|r-r'|^3}\bigg|_{r'=0}\cdot(-\vec r') + \dots =$$

$$= \frac{1}{|\vec r|} + \frac{\vec r\cdot\vec r'}{r^3} + \dots$$

$$\vec A \simeq \frac{\mu_0}{4\pi}\int \vec j(r')\left[\frac{1}{r} + \frac{\vec r\cdot\vec r'}{r^3} + \dots\right] d^3r' \Rightarrow$$

$$A_i = \frac{\mu_0}{4\pi}\left[\frac{1}{r}\int j_i(r')\,d^3r' + \frac{\vec r}{r^3}\cdot\int j_i(r')\,\vec r'\,d^3r' + \dots\right]$$

In the case of magnetostatics $\vec\nabla\cdot\vec j = 0$

$$\Rightarrow \int j_i(r')\,d^3r' = 0$$

$$\int (r_i' j_j + r_j' j_i)\,d^3r' = 0$$

$$\Rightarrow \vec r\cdot\int j_i\,\vec r'\,d^3r = \sum_j r_j\int j_i r_j'\,d^3r = \frac{1}{2}\sum_j r_j\int (j_i r_j' - j_j r_i')\,d^3r$$

$$\Rightarrow \vec r\cdot\int j_i\,\vec r'\,d^3r = \sum_j r_j \int j_i r_j'\,d^3r = \frac{1}{2}\sum_j r_j \int (j_i r_j' - j_j r_i')\,d^3r$$

$$= -\frac{1}{2}\left[\vec r \times \int (\vec r' \times \vec j)\,d^3r\right]_i$$

$$\Rightarrow \vec A \simeq -\frac{\mu_0}{8\pi}\frac{\vec r\times}{r^3}\int(\vec r'\times\vec j)\,d^3r' = \frac{\mu_0}{4\pi}\frac{\vec m\times\vec r}{r^3}, \text{ where}$$

$$m = \frac{1}{2}\int \vec r'\times\vec j(r')\,d^3r'$$

$$\vec B = \vec\nabla\times\vec A = \frac{\mu_0}{4\pi}\vec\nabla\times\left[\vec m\times\frac{\vec r}{r^3}\right] = \frac{\mu_0}{4\pi}\left[\vec m\,\vec\nabla\cdot\frac{\vec r}{r^3} - (\vec m\cdot\vec\nabla)\frac{\vec r}{r^3} + \left(\frac{\vec r}{r^3}\cdot\vec\nabla\right)\vec m - \frac{\vec r}{r^3}(\vec\nabla\cdot\vec m)\right]$$

$\propto \delta(x)$

$$m_x \frac{\partial}{\partial x}\frac{\vec r}{r^3} = m_x\left[-3\vec r\,\frac{x}{r^5}+\frac{\hat x}{r^3}\right]$$

...

$$(m\cdot\vec\nabla)\frac{\vec r}{r^3} = \frac{\vec m}{r^3} - 3\vec r\,\frac{\vec m\cdot\vec r}{r^5}$$

$$\Rightarrow \vec B = \frac{\mu_0}{4\pi}\frac{3\hat r(\vec m\cdot\hat r)-\vec m}{r^3} + \frac{2\mu_0}{3}\vec m\,\delta(r)$$

For a planar loop current $\vec m = I \times \vec A$

If a localized current distribution is placed in external magnetic field, $B(\vec r) \rightarrow$

$$B_\alpha(r)\simeq B_\alpha(0)+(\vec r\cdot\vec\nabla)B_\alpha\big|_0+\dots$$

Since $\vec{F} = \int \bar{j} \times \bar{B}\, d^3r'$

$$F_i = \sum_{jk} \epsilon_{ijk} \left[\int j_j(r')\, B_k(0)\, d^3r' + \int j_j(r') (\vec{r}' \cdot \vec{\nabla}) B_k \Big|_0 d^3r' + \dots \right]$$

$$= \sum_{jk} \epsilon_{ijk} (\bar{m} \times \vec{\nabla})_j\, \bar{B}_k(\bar{r})$$

$$\vec{F} = (\bar{m} \times \vec{\nabla}) \times \vec{B} = \vec{\nabla}(\bar{m} \cdot \bar{B}) - \vec{m}(\vec{\nabla} \cdot \bar{B}) = \vec{\nabla}(\bar{m} \cdot \bar{B})$$

total torque: $\vec{N} = \bar{m} \times \bar{B}(0)$

Since $\vec{F} = -\vec{\nabla} U$,

$$U = -\bar{m} \cdot \bar{B}$$

Macroscopic equations

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In materials, in addition to free current density there exists magnetization (from molecules, etc.)

$$\vec{M}(\vec{r}) = \sum_i N_i \langle \vec{m}_i \rangle$$

$$A(\vec{r}) \simeq \frac{\mu_0}{4\pi}\int\left[\frac{\vec{j}(r')}{|r-r'|} + \frac{m(r')\times(\vec{r}-\vec{r}')}{|r-r'|^3}\right] d^3r^3 =$$

$$= \frac{\mu_0}{4\pi}\int\left[\frac{\vec{j}(r')}{|r-r'|} + \vec{m}(r')\times\vec{\nabla}_{r'}\frac{1}{|r-r'|}\right] d^3r = \text{(by parts)}$$

$$= \frac{\mu_0}{4\pi}\int \frac{\vec{j}(r') + \vec{\nabla}'\times\vec{m}}{|r-r'|} d^3r$$

$\Rightarrow$ when averaging the equation

$$\bar{\nabla}\cdot\bar{B}_{micro} = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\bar{B} = 0$$

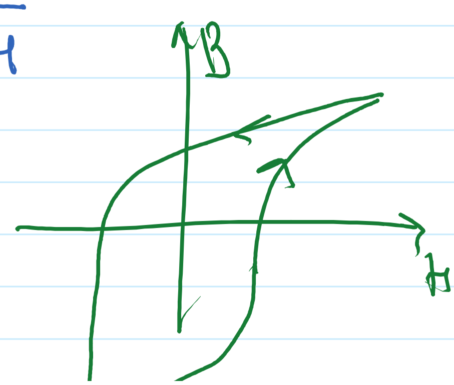
$$\bar{\nabla}\times\bar{B}_{micro} = \mu_0 j_{micro} \quad \Rightarrow \quad \bar{\nabla}\times\bar{B} = \mu_0\left[\bar{j} + \bar{\nabla}\times\bar{M}\right]$$

macroscopic magnetic field: $\vec{H} = \frac{\bar{B}}{\mu_0} - \bar{M}$

$$\bar{\nabla}\times\bar{H} = \vec{J}$$

For linear materials, $\bar{B} = \mu\bar{H}$

In ferromagnets, $\bar{B} = F(\bar{H})$:



Macroscopic eqs. would

Macroscopic eqs. yield

Boundary conditions:

$B_n = \text{const}$

$\Delta H_\tau = K \qquad \left(\bar{n} \times [\bar{H}_{2_\tau} - \bar{H}_{1_\tau}] = \vec{K} \right)$

with K - idealized surface current density

If we know $\vec{M}$ (ferromagnets) I we can use either scalar or vector potentials:

potential formulations via $\bar{\nabla} \times \bar{A} = \bar{B}$

$$\bar{\nabla} \times \bar{H} = \bar{\nabla} \times \left(\frac{\bar{B}}{\mu_0} - \bar{M} \right) = 0 \quad [J = 0]$$

in Coulomb gauge:

$$(\vec{\nabla} \cdot \vec{\nabla}) \vec{A} = -\mu_0 \vec{J}_m$$

$$\bar{A}(\bar{r}) = \frac{\mu_0}{4\pi} \int \frac{\bar{\nabla}' \times \bar{M}}{|r - r'|} d^3 r' + \underbrace{\frac{\mu_0}{4\pi} \oint \frac{M(r') \times \hat{n}'}{|r - r'|} d^2 a}_{\text{for discontinuous distributions}}$$

In the same limit:

$$\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\mu_0 H + \mu_0 M) = 0$$

$$\vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M} \quad ; \quad \vec{\nabla}\times\vec{H} = 0$$

$$-\vec{\nabla}\cdot(-\vec{\nabla}\varphi) = \Delta\varphi_M = -\rho_M, \quad \text{where } \vec{H} = -\vec{\nabla}\varphi_M ; \ \rho_M = -\vec{\nabla}\cdot\vec{M}$$

$$\Rightarrow \varphi_M(\bar{r}) = -\frac{1}{4\pi}\int\frac{\rho(r')}{|r-r'|}d^3r' = -\frac{1}{4\pi}\int\frac{\vec{\nabla}'\cdot\vec{M}(r')}{|r-r'|}d^3r' =$$

$$= \frac{1}{4\pi}\int\vec{M}(r')\cdot\vec{\nabla}'\frac{1}{|r-r'|}d^3r' = -\frac{1}{4\pi}\int M(r')\vec{\nabla}_r\frac{1}{|r-r'|}d^3r'$$

$$= -\frac{1}{4\pi}\vec{\nabla}\cdot\int\frac{M(r')}{|r-r'|}d^3r' + \underbrace{\frac{1}{4\pi}\oint\frac{\hat{n}'\cdot\vec{M}(r')}{|r-r'|}d^3r}$$

Consider, for example, uniformly magnetized sphere.

Then, $\vec{m} = \frac{4\pi R^3}{3}\vec{M}$

magnetic field: $\vec{H}_{in} = -\frac{1}{3}\vec{M}$

$$\Rightarrow \vec{B}_{in} = \frac{2\mu_0}{3}\vec{M}$$

Magnetized sphere in External field:

$$\begin{cases} H_{in} = \frac{1}{\mu_0}B_0 - \frac{1}{3}M \end{cases}$$

$$\begin{cases} H_{in} = \frac{1}{\mu_0} B_0 - \frac{1}{3} M \\ \\ B_{in} = B_0 + \frac{2}{3}\mu_0 M \end{cases}$$

if sphere is linear, $B_{in} = \mu H_{in}$

$$\Rightarrow M = 3\,\frac{\mu/\mu_0 - 1}{2\mu_0 + \mu}\, B_0$$

in general

$$B_{in} + 2\mu_0 H_{in} = 3 B_0$$