Faraday law

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Faraday discovered that

$$\mathcal{E} = -k \frac{d\Phi}{dt}$$

where $\mathcal{E} = \oint \bar{E}\cdot d\bar{l}$ is EMF

and $\Phi = \int \bar{B}\cdot d\bar{n}$ is magnetic flux

$$\Rightarrow \quad \oint \vec{E}\cdot d\vec{l} = -k \frac{d}{dt}\int \bar{B}\cdot d\bar{n}$$

In the moving frame

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}$$

$$\bar{\nabla}\times(\bar{v}\times\bar{B})\Big|_{v=const} = \vec{v}\,(\bar{\nabla}\cdot\bar{B}) - (\vec{v}\cdot\vec{\nabla})\vec{B} = -(\vec{v}\cdot\vec{\nabla})\vec{B}$$

$$\frac{d\vec{B}}{dt} = \frac{\partial\vec{B}}{\partial t} - \vec{\nabla}\times(\vec{v}\times\vec{B})$$

$$\oint \vec{E}'\cdot d\vec{l} = -k\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{n} + k\oint \vec{v}\times\vec{B}\cdot d\vec{l}$$

$$\oint \vec{E}\cdot d\vec{l} = -k\int \frac{\partial \bar{B}}{\partial t}\cdot d\bar{n}$$

where $\vec{E} = \vec{E}' - k\,\vec{v}\times\vec{B}$ $\Rightarrow$ in moving frame $\bar{E}' = \bar{E} + k\,\bar{v}\times\bar{B}$

$$\int \vec{\nabla} \times \vec{E}_1 \cdot d\bar{n} = -k \int \frac{\partial \bar{B}}{\partial t} \cdot d\bar{n}$$

$\sigma \quad \bar{E}_1' = \bar{E}_1 + k\,v \times B$

$$\vec{\nabla} \times \vec{E}_1 = -k \frac{\partial \vec{B}}{\partial t}$$

to figure out the value of the constant k, consider the moving charge

rest frame of the charge: $F = q\bar{E}_1' = kq\,\bar{v} \times \bar{B}$

lab. frame $\qquad F = q\,v \times \bar{B}$

$\Rightarrow k = 1 \quad (SI)$

$$\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \quad ; \quad \bar{E}_1' = \vec{E}_1 + \bar{v} \times \bar{B}$$

Magnetic energy

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$$\frac{dW}{dt} = -I \cdot \mathcal{E} = I \frac{d\Phi}{dt}$$

One loop: $\delta \tilde{W} = I \, \delta\Phi = \int_S \bar{j} \cdot \delta \bar{B} \cdot d\vec{n} =$

$$= \int_S j \, (\vec{\nabla} \times \delta\vec{A}) \cdot d\vec{n} = \oint \underbrace{j \, d\vec{l}}_{\vec{j}} \cdot \delta\vec{A} = \oint \vec{j} \cdot \delta\vec{A}$$

$\Rightarrow \delta W = \int d^3r \, (\vec{j} \cdot \delta\vec{A}) \leftarrow$ whole volume

Assuming linear relationship between $\vec{j}$ and $\vec{A}$,

$$\delta(\bar{j} \cdot \bar{A}) = \delta j \cdot A + \bar{j} \cdot \delta\bar{A} = 2\,\bar{j} \cdot \delta\bar{A}$$

$$\Rightarrow \quad W = \frac{1}{2} \int \vec{j} \cdot \vec{A} \, d^3r$$

Alternatively,

$$\delta W = \int \delta A \cdot (\bar{\nabla} \times \bar{H}) \, d^3r$$

$$\vec{\nabla} \cdot (\bar{A} \times \bar{B}) = \vec{B} \cdot (\bar{\nabla} \times \bar{A}) - \vec{A} \cdot (\bar{\nabla} \times \vec{B})$$

$$\delta W = \int \bar{H} \cdot (\bar{\nabla} \times \delta\bar{A}) \, d^3r + \int \bar{\nabla} \cdot (\bar{H} \times \delta\bar{A}) \, d^3n =$$

$$= \int \vec{H} \cdot \delta\vec{B} \, d^3r + \cancel{\oint_S \bar{H} \times \delta\bar{A} \, d^2a}^{\,\to 0}$$

For linear materials,

$$\delta W = \frac{1}{2} \int \delta(\vec{H} \cdot \vec{B})\, d^3 r$$

$$W = \frac{1}{2} \int \vec{H} \cdot \vec{B}\, d^3 r$$