Displacement current

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Up ~~until~~ now the equations governing EM response look like:

(7.1)
$$\begin{cases} \operatorname{div} \boxed{\vec{E}} = \frac{\rho}{\epsilon_0} \\ \operatorname{div} \bar{B} = 0 \\ \operatorname{curl} \bar{H} = \vec{j} \qquad \Rightarrow \quad \bar{\nabla} \cdot \bar{j} = \bar{\nabla} \cdot \left(\bar{\nabla} \times \bar{H} \right) = 0 \\ \operatorname{curl} \vec{E} = -\frac{\partial B}{\partial t} \end{cases}$$

However, moving charge represents current,

$$\frac{\partial \rho}{\partial t} + \bar{\nabla} \cdot \bar{j} = 0 \quad \Rightarrow \text{ the equations } (7.1)$$

are not compatible with electrodynamics.

to make them compatible, Maxwell introduced displacement current

$$\vec{j}_d = \frac{\partial \bar{D}}{\partial t} = \frac{\partial}{\partial t} \epsilon_0 \epsilon \vec{E} = \frac{\partial}{\partial t} \epsilon_0 \vec{E} + \frac{\partial \bar{P}}{\partial t}$$

full Maxwell eqs become:

$$\operatorname{div} \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\operatorname{div} \vec{B} = 0$$

(7.1a)

(7.1a) $\begin{cases} \operatorname{div} \vec{B} = 0 \\ \operatorname{curl} \vec{H} = \dfrac{\partial \vec{D}}{\partial t} + \vec{j} \\ \operatorname{curl} \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \end{cases}$ $\Rightarrow \vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = 0 = \dfrac{\partial}{\partial t}\vec{\nabla}\cdot\vec{D} + \vec{\nabla}\cdot\vec{j} = \dfrac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j}$

As before, we can introduce potentials $V, \vec{A}$

where $\boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$ $[\Leftarrow \operatorname{div} B = \operatorname{div} \operatorname{curl} \vec{A} = 0]$

now $\operatorname{curl} \vec{E} = -\dfrac{\partial}{\partial t}(\operatorname{curl} \vec{A}) \Rightarrow$

$$\operatorname{curl}\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

$$\Rightarrow \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V$$

$$\boxed{\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}}$$

Maxwell eqs in potentials (in vacuum) becomes

$$\begin{cases} \dfrac{1}{\mu_0}\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \epsilon_0 \dfrac{\partial}{\partial t}\left(-\vec{\nabla} V - \dfrac{\partial \vec{A}}{\partial t}\right) + \vec{j} \\ \vec{\nabla}\cdot\left(\vec{\nabla} V + \dfrac{\partial \vec{A}}{\partial t}\right) = -\dfrac{\rho}{\epsilon_0} \end{cases}$$

(others eqs are automatically satisfied)

$$\frac{1}{\mu_0}\left[\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \Delta\vec{A}\right] = -\epsilon_0 \vec{\nabla}\frac{\partial V}{\partial t} - \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} + \vec{j}$$

(7.2) $$\begin{cases} \nabla^2 V + \dfrac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = -\dfrac{\rho}{\epsilon_0} \\ \nabla^2\vec{A} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \dfrac{1}{c^2}\dfrac{\partial V}{\partial t}\right) - \dfrac{1}{c^2}\dfrac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{j} \end{cases}$$

Note that the equations don't change under Gauge transformations:

(7.3) $$\boxed{\vec{A} \to \vec{A} + \vec{\nabla}\Lambda \; ; \; V \to V - \frac{\partial \Lambda}{\partial t}}\,,$$ where

Λ - arbitrary function.

$\Rightarrow$ Lorentz gauge: $\vec{\nabla}\cdot A + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$

(7.4) $\Rightarrow$ M. eq:

$$\left.\begin{cases} \nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t} = -\frac{\rho}{\epsilon_0} \\ \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{j} \end{cases}\right\}$$ two wave eqn

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Coulomb gauge:

$$\vec{\nabla}\cdot\vec{A} = 0$$

(7.5)

$$\begin{cases} \nabla^2 \varphi = -\rho/\epsilon_0 \Rightarrow \varphi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}\,d^3r' \\ \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \underbrace{-\mu_0 \vec{j} + \frac{1}{c^2}\vec{\nabla}\frac{\partial \varphi}{\partial t}}_{-\mu_0 \vec{J}_{eff}} \end{cases}$$

"instantaneous" Coulomb solution

$\Leftarrow$ inhomogeneous wave eqn

Any vector can be written as a sum of

"longitudinal" (irrotational) vector: $\vec{\nabla}\times J_l = 0$

"transverse" (solenoidal) vector: $\vec{\nabla}\cdot J_t = 0$

Eq. (7.5c) becomes:

(7.6) $$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{j}_t$$

(7.6) $$\begin{cases} \nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\mu_0 j_t \\ \vec{\nabla}\frac{\partial \varphi}{\partial t} = \frac{1}{\epsilon_0}\frac{\partial \vec{j}_\ell}{\partial t} \end{cases}$$

Note that when no sources are present,

(7.7) $$\begin{cases} \varphi = 0 \;;\; \nabla^2 \vec{A} = \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} \;; \\ \vec{E} = -\frac{\partial \vec{A}}{\partial t} \;;\; \vec{B} = \vec{\nabla}\times\vec{A} \end{cases}$$

Green function for wave equation

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Consider wave equations:

(7.7) $$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = -4\pi f(\vec{r},t)$$

in free space (no boundary surfaces);

In Fourier domain

(7.8) $$\begin{cases} \psi(\vec{r},t) = \frac{1}{2\pi}\int \psi(\vec{r},\omega) e^{-i\omega t} d\omega \\ \psi(\bar{r},\omega) = \int \psi(r,t) e^{i\omega t} dt \end{cases}$$

$\Rightarrow$ Eq. (7.7) becomes:

(7.9) $$\nabla^2 \psi(\bar{r},\omega) + \frac{\omega^2}{c^2}\psi(\bar{r},\omega) = -4\pi f(\vec{r},\omega)$$

The latter eq. can be solved using Green's function formalism

G.F. eqn:

(7.10) $$(\nabla^2 + k^2) G_k(r,r') = -4\pi\delta(\bar{r}-\bar{r}')$$

from symmetry consideration $G_k(\bar{r},\bar{r}') = G_k(\bar{R})$,

where $R = \bar{r}-\bar{r}'$

(7.10a) $$\Rightarrow \frac{1}{R}\frac{d^2}{dR^2}(RG_k) + k^2 G_k = -4\pi\delta(\bar{R})$$

① $R \neq 0$: $$\frac{1}{R}\frac{d^2}{dR^2}(RG_k) + k^2 G_k = 0$$

$$\Rightarrow \frac{d^2}{dR^2}(RG_k) = -k^2(RG_k) \Rightarrow$$

$$G_k = \frac{1}{R}\left(Ae^{ikR} + Be^{-ikR}\right) = AG_k^+(R) + BG_k^-(R)$$

(7.11) $$\boxed{G_k^\pm = \frac{e^{\pm ikR}}{R}}$$

In the limit $R \to 0$: $\frac{1}{R}\frac{d^2}{dR^2}(RG_k) \simeq -4\pi\delta(R) \Rightarrow G \sim \frac{1}{R}$

To figure out A, B constants, look at time dependence:

$$G^\pm(R, t-t') = \frac{1}{2\pi}\int G_k(R,\omega)e^{-i\omega(t-t')}\,d\omega$$

$$\left(\nabla_r^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G^\pm = \int\left(\nabla_r^2 + \frac{\omega^2}{c^2}\right)G(R,\omega)\frac{e^{-i\omega(t-t')}d\omega}{2\pi} =$$

$$= -4\pi\,\delta(\vec{R})\int\frac{e^{-i\omega(t-t')}d\omega}{2\pi} = -4\pi\,\delta(\bar{r}-\bar{r}')\,\delta(t-t')$$

$\Rightarrow$ $G^\pm(R, t-t')$ satisfies G. F. for wave eqn. We can calculate $G(R,\tau)$ explicitly:

(7.12) $$G^\pm(R,t) = \frac{1}{2\pi}\int\frac{1}{R}e^{\pm ikR - i\omega(t-t')}\,d\omega =$$

$$= \frac{1}{2\pi R}\int e^{-i\omega(t-t'\mp\frac{R}{c})}\,d\omega = \frac{1}{R}\,\delta\left(\left[t\mp\frac{R}{c}\right]-t'\right)$$

$G^+(R,\tau) \to$ retarded Green fn (causal)

$G^-(R,\tau) \to$ advanced G. F (used when $\psi(r, t\to\infty)$ known)

$\Rightarrow$ Solution to wave equ:

$$(7.13)\quad \psi(\vec{r},t) = \psi_h(\bar{r},t) + \iint f(r',t')\, G^+(\bar{r}-\bar{r}', t-t')\, d^3\vec{r}'\, dt' =$$

$$= \psi_h(\bar{r},t) + \int \frac{\left[f(\bar{r},t')\right]_{ret}}{|\bar{r}-\bar{r}'|}\, d^3\bar{r}'$$

$$t' = t - \frac{|\bar{r}-\bar{r}'|}{c}$$

Poynting theorem

Monday, October 10, 2016 9:23 PM

$$\bar{\nabla} \times H = \frac{\partial \bar{D}}{\partial t} + \vec{j}$$

The total work done by the fields:

$$(2.14) \quad \int \bar{J} \cdot \bar{E}\, d^3 r = \int \left(\bar{\nabla} \times \bar{H} - \frac{\partial \bar{D}}{\partial t} \right) \cdot \vec{E}\, d^3 r =$$

$$= \int \left[\bar{E} \cdot (\bar{\nabla} \times \bar{H}) - \frac{\partial \bar{D}}{\partial t} \cdot \vec{E} \right] d^3 r$$

$$(2.15) \quad \vec{\nabla} \cdot (\bar{E} \times \bar{H}) = \vec{H} \cdot (\bar{\nabla} \times \bar{E}) - \bar{E} (\bar{\nabla} \times \bar{H})$$

$$\Rightarrow \int \bar{J} \cdot \bar{E}\, d^3 r = - \int \left[\underbrace{\bar{H} \cdot \left(\frac{\partial \bar{B}}{\partial t} \right) + \bar{\nabla} \cdot (\bar{E} \times \bar{H}) + \bar{E} \cdot \frac{\partial \bar{D}}{\partial t}}_{\frac{\partial u}{\partial t},\ u = \frac{1}{2} (\bar{E} \cdot \bar{D} + \bar{B} \cdot \bar{H})} \right] d^3 r$$

$$(2.16) \Rightarrow \boxed{\frac{\partial u}{\partial t} + \vec{\nabla} \cdot (\bar{E} \times \bar{H}) = -\bar{J} \cdot \bar{E}}, \quad \leftarrow \text{conservation law}$$

$$(2.16a) \quad \boxed{\vec{S} = \bar{E} \times \bar{H}} \text{ — Poynting vector; energy flux}$$

dispersion to be discussed later

In the presence of losses,

$$E(\vec{r}, t) = \frac{1}{2\pi} \int E(\bar{r}, \omega) e^{-i\omega t}\, d\omega$$

$$\vec{E} \cdot \frac{\partial \bar{D}}{\partial t} = \mathrm{Re} \left[\int d\omega \int d\omega' \frac{1}{4\pi^2} \bar{E}(\bar{r}, \omega') \epsilon(\omega) (-i\omega) \vec{E}(\vec{r}, \omega) e^{-i(\omega+\omega')t} \right]$$

$$\mathrm{Re} \left[\int \int \frac{1}{\ldots} \ldots \bar{E}(\ldots, \omega') (\ldots) \epsilon(\omega) \vec{E}(\ldots) e^{-i(\omega-\omega')t} \right]$$

$$= \mathrm{Re}\left[\iint d\omega\, d\omega' \frac{1}{4\pi^2} \vec{E}_1(r, -\omega')(-i\omega\,\epsilon(\omega))\vec{E}_1(r,\omega) e^{-i(\omega-\omega')t}\right] =$$

$$= \frac{1}{2}\iint \frac{d\omega\, d\omega'}{4\pi^2} \vec{E}_1^*(r, \omega')\underbrace{\left(i\omega'\epsilon^*(\omega') - i\omega\epsilon(\omega)\right)}\vec{E}_1(r,\omega) e^{-i(\omega-\omega')t}$$

when $\omega' \simeq \omega$

$$\longrightarrow 2\omega\, \mathrm{Im}(\epsilon) - i(\omega-\omega')\frac{d}{d\omega}(\omega\epsilon^*(\omega)) + \dots$$

$$(7.17)\begin{cases} \left\langle \vec{E}_1 \cdot \frac{\partial \vec{D}}{\partial t} \right\rangle = \omega_0\, \mathrm{Im}(\epsilon_0) \langle E_1^2 \rangle + \frac{\partial u^{(e)}_{eff}}{\partial t} \\ u^{(e)}_{eff} = \frac{1}{2} \mathrm{Re}\left[\frac{d(\omega\epsilon)}{d\omega}\right] \langle E_1^2 \rangle \end{cases}$$