1. Consider the differential equation \[ \frac{dx}{dt} = -x(x - 2)^2. \]
You are not expected to solve this DE explicitly (nor should you), in order to answer the questions below.
(a) Find the critical points of this DE
\[ x = 0, 2 \] are the two c.p.'s; Only \( x = 0 \) is stable.
(b) Sketch the phase line for this DE
(c) Determine the stability of the critical points
(d) Carefully sketch the 4 solution curves \( x = x(t) \) for the initial conditions \( x(0) = 0, x(0) = 2, x(0) = 3, x(0) = 4. \)
(e) With the help of the phase line, and the solution curves above, determine the \( \lim x(t) \) as \( t \to \infty \) if \( x(0) = 3. \)
\[ \lim x(t) = 2. \]

2. Solve the IVP \[ x'' + 4x' + 8x = 0; \quad x(0) = 0, x'(0) = 2 \]
\( \text{Char Poly} \) \( (r^2 + 4r + 8) = 0 \) has \( r = -2 \pm 2i \) as roots. So, \( x(t) = e^{-2t}(c_1 \cos 2t + c_2 \sin 2t) \). IC's imply \( c_1 = 0, c_2 = 1, \) and thus \( x(t) = e^{-2t} \sin 2t \)

3. Given the initial value problem \[ \frac{dy}{dx} = -xy + y^2, y(0) = 1. \]
Use Euler’s numerical algorithm with step size \( h = 0.1 \) to estimate \( y(0.1) \) and \( y(0.2) \).
\[ y(0.1) \approx y(0) + (0.1)f(0, 1) = 1 + (0.1)1 = 1.1 = y_1. \] Since \( f(x_1, y_1) = f(1, 1.1) = -1.1 + 1.21 = 1.1, \) then \( y(0.2) \approx y_2 = y_1 + 0.1f(x_1, y_1) = 1.1 + (0.1)1.1 = 1.21. \)

4. Suppose that the population \( P(t) \) (in millions) of a country satisfies the differential equation
\[ \frac{dP}{dt} = kP(200 - P) \]
with \( k \) constant. Its population in 1940 was 100 million and was then growing at a rate of 1 million per year. In which year will the population exceed 160 million?
At \( t = 0, P'(0) = 1 = kP(0)(200 - P(0)) = k10^4, \) giving \( k = 10^{-4}. \) Thus \( \frac{dP}{P(200 - P)} = 10^{-4}dt. \)
Since \( \frac{1}{P(200 - P)} = (1/200)[\frac{1}{2} + \frac{1}{200 - P}], \) we have \( (1/200)[\ln P - \ln(200 - P)] = 10^{-4}t + C. \) Setting \( P(0) = 100, \) yields \( C = 0. \) Solving for \( t = 50 \ln \frac{P}{200- P}. \) Thus \( P = 160 \) when \( t = 50 \ln 4 \approx 69.14. \) Thus early in the year 2009, the population will exceed 160 million for the first time.

5. Consider the following differential equations for \( u(t) \) the displacement of a spring from its relaxed length at time \( t. \)
(a) The unforced spring satisfies the homogenous DE

\[ u'' + u' + 4u = 0 \]

Find the solution for the unforced spring.

\[ u = e^{-t/2}[c_1 \cos(\sqrt{15}t/2) + c_2 \sin(\sqrt{15}t/2)] \]

(b) A motor is turned on and drives the spring so that \( u \) satisfies

\[ u'' + u' + 4u = \sin t \]

Verify that \( u_p(t) = (-1/10) \cos t + (3/10) \sin t \) is a particular solution to this forced spring system.

(c) Write down the general solution to forced spring system

\[ u'' + u' + 4u = \sin t. \]

\[ u = e^{-t/2}[c_1 \cos(\sqrt{15}t/2) + c_2 \sin(\sqrt{15}t/2)] - (1/10) \cos t + (3/10) \sin t \]