92.236.204
Engineering DE's – Test 2 Spring 09 **Name:** ______________________

Box in your answer to each question. Please fold your exam into the test booklet. Good luck.

1. Consider the differential equation
$$\frac{dx}{dt} = -x(x-2)^2.$$
You are not expected to solve this DE explicitly (nor should you), in order to answer the questions below.
   (a) Find the critical points of this DE
   $x = 0, 2$ are the two c.p.'s; Only $x = 0$ is stable.
   (b) Sketch the phase line for this DE
   (c) Determine the stability of the critical points
   (d) Carefully sketch the 4 solution curves $x = x(t)$ for the initial conditions $x(0) = 0$, $x(0) = 2$, $x(0) = 3$, $x(0) = 4$.
   (e) With the help of the phase line, and the solution curves above, determine the $\lim x(t)$ as $t \to \infty$ if $x(0) = 3$.
   $\lim x(t) = 2$.

2. Solve the IVP
$$x'' + 4x' + 8x = 0; \;\; x(0) = 0, x'(0) = 2$$
Char Poly $(r^2 + 4r + 8) = 0$ *has* $r = -2 \pm 2i$ *as roots. So,* $x(t) = e^{-2t}(c_1 \cos 2t + c_2 \sin 2t)$. *IC's imply* $c_1 = 0, c_2 = 1$, *and thus* $x(t) = e^{-2t} \sin 2t$

3. Given the initial value problem
$$\frac{dy}{dx} = -xy + y^2, y(0) = 1.$$
Use Euler's numerical algorithm with step size $h = 0.1$ to estimate $y(0.1)$ and $y(0.2)$.
$y(0.1) \approx y(0) + (0.1)f(0,1) = 1 + (.1)1 = 1.1 = y_1$, *Since* $f(x_1, y_1) = f(.1, 1.1) = -.11 + 1.21 = 1.1$, *then* $y(0.2) \approx y_2 = y_1 + 0.1f(x_1, y_1) = 1.1 + (.1)1.1 = 1.21$.

4. Consider the following differential equations for $u(t)$ the displacement of a spring from its relaxed length at time t.
   (a) The unforced spring satisfies the homogenous DE
$$u'' + u' + 4u = 0$$
   Find the solution for the unforced spring.
   $u = e^{-t/2}[c_1 \cos(\sqrt{15}t/2) + c_2 \sin(\sqrt{15}t/2)]$
   (b) A motor is turned on and drives the spring so that u satisfies
$$u'' + u' + 4u = \sin t$$
   Verify that $u_p(t) = (-1/10)\cos t + (3/10)\sin t$ is a particular solution to this forced spring system.

(c) Write down the general solution to forced spring system

$$u'' + u' + 4u = \sin t.$$

$u = e^{-t/2}[c_1 \cos(\sqrt{15}t/2) + c_2 \sin(\sqrt{15}t/2)] - (1/10)\cos t + (3/10)\sin t$

5. In a town of $100,000$ persons, a malicious rumor about the intelligence of engineers is spreading so that $N(t)$, the number of people who have heard the rumor after t days, is growing so that

$$\frac{dN}{dt} = kN(100,000 - N)$$

where k is constant. Suppose that at time $t = 0$, half of the population have heard the rumor and that the number of those who have heard it is then increasing at the rate of 1000 persons per day. On which day will the rumor have spread to 80% of the population?

Substituting $t = 0$ *into the DE and using* $N'(0) = 1000$, *and* $N(0) = 50,000$ *gives* $10^3 = k(5 \times 10^4)^2$. *Solving we have* $k = 4 \times 10^{-7}$. *The DE is separable and* $\frac{dN}{N(100,000-N)} = kdt$ *so since* $\frac{1}{N(100,000-N)} = 10^{-5}[\frac{1}{N} + \frac{1}{100,000-N}]$. *Thus integrating, we get* $10^{-5}[\ln N - \ln(100,000 - N)] = kt + C$. *At* $t = 0$ *we get* $C = 0$ *so substituting for* k *gives* $\ln \frac{N}{10^5 - N} = 4 \times 10^{-2}t$. *Setting* $N = 80,000$ *and solving for* $t = (1/4)10^2 \ln 4$ *yield* $t = 50 \ln 2 \approx 34.5$ *days. Thus some time after noon on the 34th day is the first time that 80% of the population will have heard the vicious rumor about intelligent engineers.*