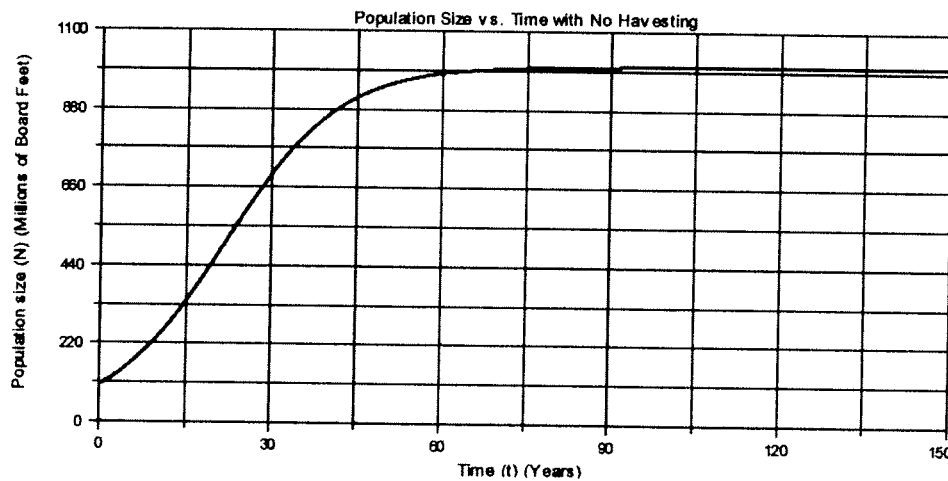


## Exploration 9.2

Before harvesting lumber from a forest, it is important to examine its growth rate and harvest rate to examine the effects that harvesting will have of the tree population in the long run. If too many trees are harvested, the company will make a lot of money at first, but the overall population will decline until there are no more trees to harvest and the company will go bankrupt. If too few trees are harvested the growth of the tree population will be very strong, but the company will not make much money. The growth rate of a forest under proportional harvesting can be modeled with the differential equation  $N' = rN \left(1 - \frac{N}{K}\right) - hN$  where  $r$  is the growth rate,  $N$  is the population size,  $K$  is the carrying capacity of the forest, and  $h$  is the harvesting rate. The quantity  $hN$  is the amount of trees that are actually harvested and is the total yield of the harvest.

### 1. No Harvesting

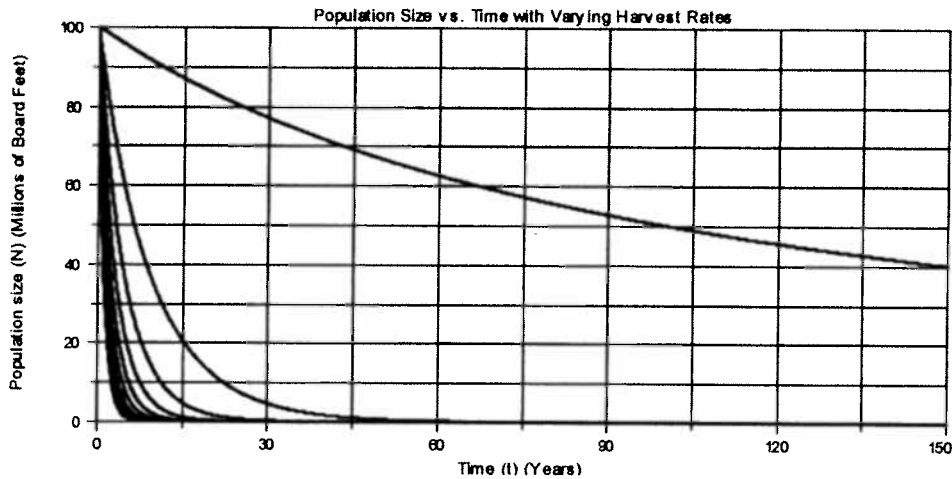
The first step in the model of the forest population is to see how the forest population behaves when there is no harvesting. To accomplish this task we can simply set the harvest rate,  $h$ , to 0 in the above equation. This gives the equation  $N' = rN \left(1 - \frac{N}{K}\right)$ . Setting  $r=0.1$  and  $K=1000$  million board feet with an initial population of 100 million board feet at  $t=0$  produces the graph below.



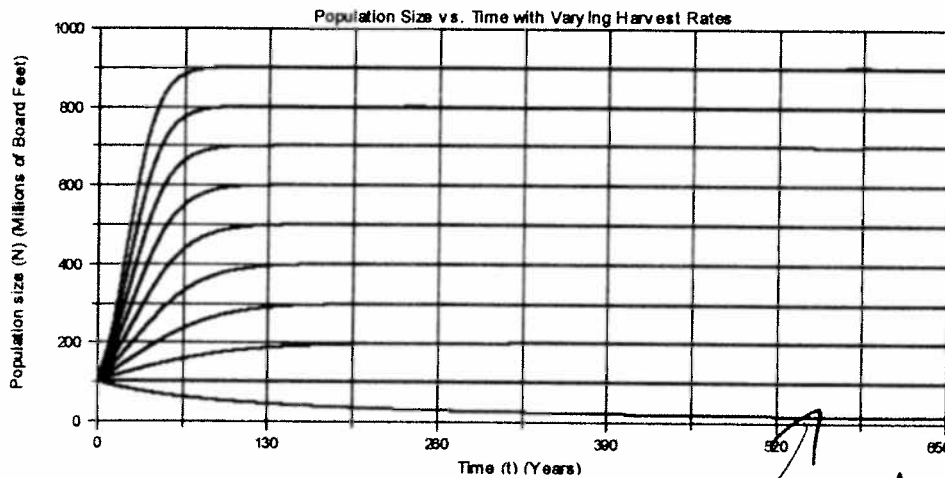
The population grows steadily without harvesting, and after a period of about 90 years, the population size has grown to almost its carrying capacity of 1000 million board feet. Since there's no harvesting, the population will not decrease at all and will continue to slowly increase towards its carrying capacity. Once it reaches it, the value of  $N/K$  will be 1, so  $(1-N/K)$  will equal zero and there will be no more change in the population size.

## 2. Varying Harvest Rates

To examine the effect that harvesting has on the population of the forest, several solutions can be graphed together, each with the same initial population, growth rate, and carrying capacity, but with different values for the harvesting rate. Since the harvesting rate is a percent, it cannot exceed 1. A value of 1 means the entire forest is harvested, since 1 would be equal to 100% of the total population. There would be a high yield for one year, but there would be no more trees after that first year. Using the values of  $r$ ,  $K$ , and  $N_0$  from the equation in the previous exploration ( $r=0.1$ ,  $K=1000$ ,  $N_0=100$ ), a sweep of the harvest rate,  $h$ , was done from 0.1 to 1.0 in increments of 0.1, producing the graph below.



The graph above shows that even with a harvest rate of 10% there is no growth in the population of the forest. It steadily declines from its initial value of 100 million board feet and will continue to decline until there are no board feet left. This makes sense, as the growth rate is at 0.1. If all the trees produced are harvested, the population will not be able to grow at all. All the other values in the sweep are larger than 0.1, so harvesting at those increased rates will just cause the population to drop off to zero sooner. A second sweep of values for  $h$  is shown below, with values ranging from 0.01 to 0.1 in increments of 0.01.



Express  
 $N(t)$   
 as a  
 function  
 of  $h$ !

which  
 $h$ 's?

Values of 1% to 8% allow the population to grow, a value of 9% keep the population at a constant 100 million board feet, and a value of 10% cause the population to decline. The goal of harvesting is to make money, and the money that the company makes is based on the yield, which is the harvest rate times the population size. While a high harvest rate will make money more quickly than a lower harvest rate, it will also limit the size of the forest, thereby reaching its maximum yield relatively quickly. A low harvest rate will not make much money at first, but it will not limit the size of the population as much as a high harvest rate will, so the increase in the population size will increase the yield, so a lower harvest rate will make more money in the long run than a higher harvest rate.

The graph makes it clear that the effective carrying capacity is lowered by the harvesting. The original carrying capacity was 1000 million board feet, and with increasing harvest rates, the carrying capacity is lowered. For a harvest rate of .01, the effective carrying capacity is lowered by 100 million board feet. When the rate is increased to .02, the effective carrying capacity is lowered by 200 million board feet. In fact, the carrying capacity which is a stable equilibrium point can be found by setting  $N' = rN \left(1 - \frac{N}{K}\right) - hN$  equal to zero and solving for N. One solution is  $N=0$ , but the other solution is found to be  $N = K \left(1 - \frac{h}{r}\right)$ . When there's no harvesting, the two equilibrium points are when  $N=0$  and when  $N=K$ , so this new value of  $K \left(1 - \frac{h}{r}\right)$  can be said to be the effective carrying capacity.

good

### 3. Maximizing Yield

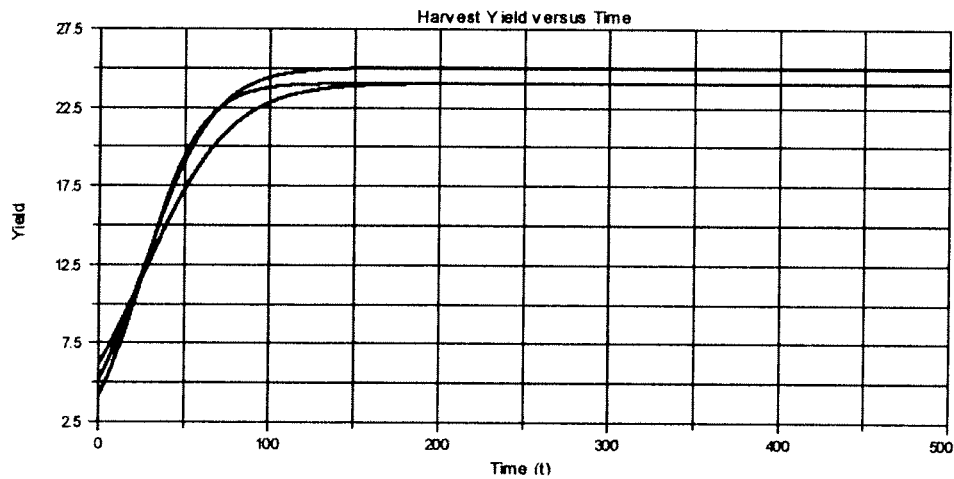
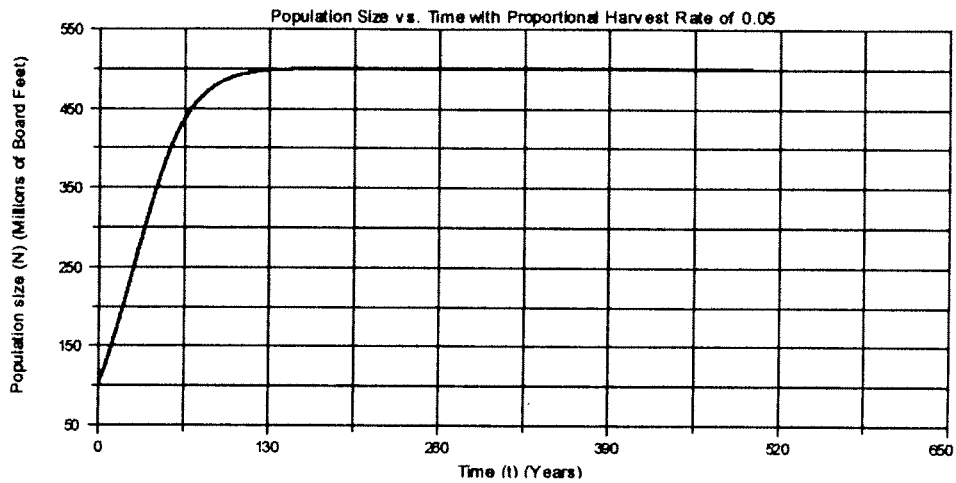
The maximum yield is really only concerned with what happens in the long run. As was shown in the previous examples, the equilibrium point is affected by the harvest rate and the yield is also affected by the harvest rate. If too many fish are harvested, the money will be great in the beginning but the population, and therefore the yield, will reach a maximum value that is lower than if less fish are harvested. On the other hand, if too little fish are harvested, the population size might be great, but because the harvest is so small, the yield is not very big even with the larger population size. It was shown that as time goes to infinity the population goes to its carrying capacity. It was also shown that the carrying capacity of the fish is altered by a factor of  $1 - \frac{h}{r}$ . Since the yield is the harvest rate times the population size, and the population size as time goes to infinity is  $K \left(1 - \frac{h}{r}\right)$ , the yield can be expressed as  $Yield = hK \left(1 - \frac{h}{r}\right)$ .

To maximize the yield, the derivative of the yield with respect to the harvesting rate can be taken and set to zero. The value of h could then be solved for. The function for yield as a function of harvest rate is  $Yield = hK - \frac{h^2K}{r}$  and the derivative is  $\frac{dY}{dh} = K - \frac{2hK}{r}$ . Setting the derivative equal to zero, the value of h that will maximize yield is when  $h=r/2$ . Since  $r=0.1$  in this model, the optimum harvest rate for long term yield is when  $h=0.05$ .

This could also be solved without taking a derivative with symmetry. Since  $Yield = hK \left(1 - \frac{h}{r}\right)$ , the yield is 0 when  $h=0$  and when  $h=r$ . The shape of a Yield vs. Harvest rate graph is a parabola, with  $h=0$  and  $h=r$  its roots. The maximum then occurs at the midpoint, when  $h=r/2$ .

good

A graph of the population size as a function of time is shown with the values  $r=0.1$ ,  $K=1000$ ,  $h=0.05$ , and  $N_0=100$ . A graph of yield as a function of time with those same values is also shown, and the additional values of  $h=0.04$  and  $h=0.06$  are on that same graph to show that  $h=0.05$  does produce the maximum long term yield.



Notice that in the yield vs. time graph, the values of  $h=0.04$  and  $h=0.06$  produce the same long term yield, but a harvest rate of 0.06 will reach its maximum yield faster than a rate of 0.04. Since the graph of yield vs. harvest rate is a parabola with its center at  $r/2$ , by symmetry the values of  $h=0.07$  will produce the same long term yield as  $h=0.03$ ;  $h=0.08$  will be the same as  $h=0.02$  and so on.