

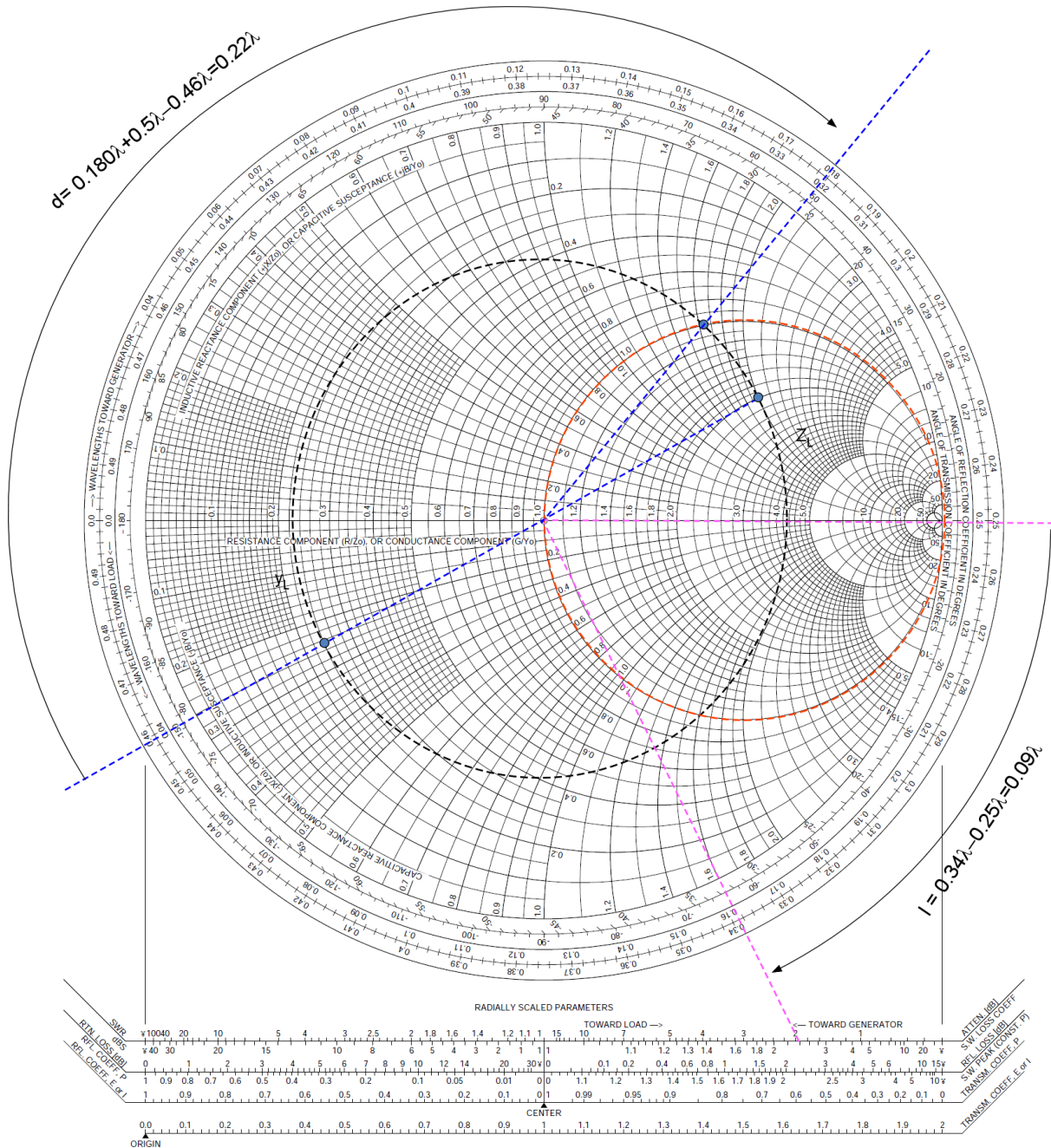
EECE.3600 Exam II

11/13/2017

Name: _____

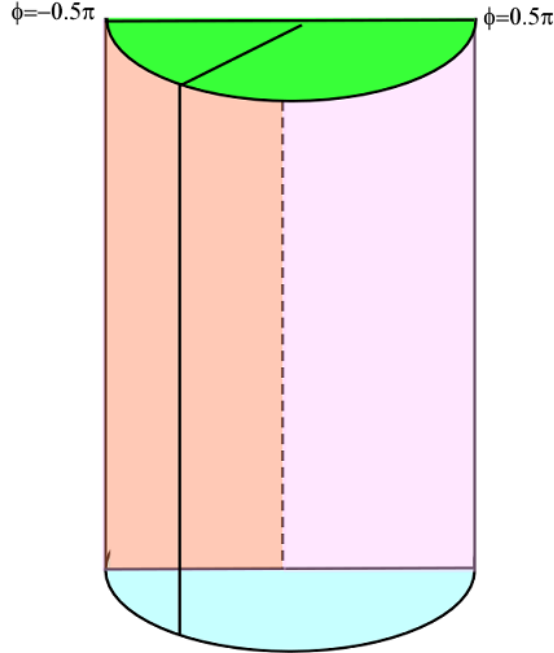
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1. (35%) On a lossless $50\text{-}\Omega$ transmission line terminated with a $Z_L = 100 + j100\ \Omega$. If this transmission line is to be matched to the load using a shorted load stub. Determine the stub length and distance between the load and stub. Two possible answers. You only need to show one of them.



2. (30%) For a vector field $\vec{A} = r^2 \hat{r} + 3r\phi \hat{\phi} - 2\hat{z}$, verify the divergence theorem

$$\oiint_s \vec{A} \cdot d\vec{s} = \iiint_v (\nabla \cdot \vec{A}) dv, \text{ on a section of a cylinder bounded by } r=1, -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}, 1 \leq z \leq 3.$$



Solution:

$$\oiint_s \vec{A} \cdot d\vec{s} = \iint_{top} \vec{A} \cdot d\vec{s} + \iint_{bottom} \vec{A} \cdot d\vec{s} + \iint_{side} \vec{A} \cdot d\vec{s} + \iint_{left} \vec{A} \cdot d\vec{s} + \iint_{right} \vec{A} \cdot d\vec{s},$$

$$\iint_{top} \vec{A} \cdot d\vec{s} = \iint_{top} -2\hat{z} \cdot \hat{z} r dr d\phi = -\pi, \quad \iint_{bottom} \vec{A} \cdot d\vec{s} = \iint_{bottom} -2\hat{z} \cdot (-\hat{z}) r dr d\phi = \pi,$$

$$\iint_{side} \vec{A} \cdot d\vec{s} = \iint_{side} r^2 \hat{r} \cdot \hat{r} r d\phi dz \Big|_{r=1} = 2\pi,$$

$$\iint_{left} \vec{A} \cdot d\vec{s} = \iint_{left} 3r\phi \hat{\phi} \cdot \hat{\phi} dr dz \Big|_{\phi=\pi/2} = \frac{3\pi}{2},$$

$$\iint_{right} \vec{A} \cdot d\vec{s} = \iint_{right} 3r\phi \hat{\phi} \cdot (-\hat{\phi}) dr dz \Big|_{\phi=-\pi/2} = \frac{3\pi}{2},$$

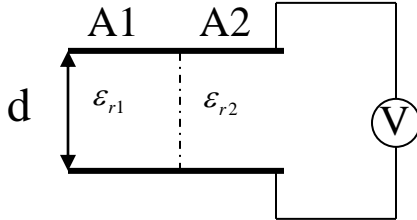
$$\oiint_s \vec{A} \cdot d\vec{s} = \iint_{top} \vec{A} \cdot d\vec{s} + \iint_{bottom} \vec{A} \cdot d\vec{s} + \iint_{side} \vec{A} \cdot d\vec{s} + \iint_{left} \vec{A} \cdot d\vec{s} + \iint_{right} \vec{A} \cdot d\vec{s} = 5\pi.$$

$$\nabla \cdot \vec{A} = \frac{1}{r} \frac{\partial}{\partial r} (r r^2) + \frac{1}{r} \frac{\partial}{\partial \phi} (3r\phi) = 3r + 3,$$

$$\iiint_v (\nabla \cdot \vec{A}) dv = \int_1^3 dz \int_{-\pi/2}^{\pi/2} d\phi \int_0^1 (3r + 3) r dr = 2\pi \left(r^3 + \frac{3}{2} r^2 \right) \Big|_0^1 = 5\pi,$$

$$\oiint_s \vec{A} \cdot d\vec{s} = \iiint_v (\nabla \cdot \vec{A}) dv.$$

3. (35%) A parallel-plate capacitor has two dielectric filling regions between the plate, $A_1 = 10\text{mm}^2$, $A_2 = 20\text{mm}^2$, $d = 5\text{mm}$, $V = 10\text{V}$ $\epsilon_{r1} = 1.5$, $\epsilon_{r2} = 3$, ignore fringing field effect. $\epsilon_0 = 8.85 \times 10^{-12} \text{ F/m}$.



Determine:

- (a) Electric field E_1 , E_2 of the two regions.
 (b) The capacitance of the capacitor.

Solution:

$$\vec{E}_1 = \vec{E}_2 = \frac{10\text{V}}{5\text{mm}} = 2 \times 10^3 (\text{V/m}),$$

$$\vec{D}_1 = \epsilon_1 \vec{E}_1, \vec{D}_2 = \epsilon_2 \vec{E}_2.$$

$$Q_1 = A_1 D_1, Q_2 = A_2 D_2$$

$$C = \frac{Q_1 + Q_2}{V} = \frac{\epsilon_1 E_1 A_1 + \epsilon_2 E_2 A_2}{E_1 d} = \epsilon_0 \frac{1.5 \cdot 10\text{mm}^2 + 3 \cdot 20\text{mm}^2}{5\text{mm}} = 0.13\text{pF}.$$

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation, $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of \mathbf{A}, $\mathbf{A} =$	$\sqrt{A_x^2 + A_y^2 + A_z^2}$	$\sqrt{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1,$ for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1,$ for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1,$ for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}} \cdot \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product, $\mathbf{A} \cdot \mathbf{B} =$	$A_x B_x + A_y B_y + A_z B_z$	$A_r B_r + A_\phi B_\phi + A_z B_z$	$A_R B_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product, $\mathbf{A} \times \mathbf{B} =$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length, $d\mathbf{l} =$	$\hat{\mathbf{x}} dx + \hat{\mathbf{y}} dy + \hat{\mathbf{z}} dz$	$\hat{\mathbf{r}} dr + \hat{\boldsymbol{\phi}} r d\phi + \hat{\mathbf{z}} dz$	$\hat{\mathbf{R}} dR + \hat{\boldsymbol{\theta}} R d\theta + \hat{\boldsymbol{\phi}} R \sin\theta d\phi$
Differential surface areas	$ds_x = \hat{\mathbf{x}} dy dz$ $ds_y = \hat{\mathbf{y}} dx dz$ $ds_z = \hat{\mathbf{z}} dx dy$	$ds_r = \hat{\mathbf{r}} r d\phi dz$ $ds_\phi = \hat{\boldsymbol{\phi}} dr dz$ $ds_z = \hat{\mathbf{z}} r dr d\phi$	$ds_R = \hat{\mathbf{R}} R^2 \sin\theta d\theta d\phi$ $ds_\theta = \hat{\boldsymbol{\theta}} R \sin\theta dR d\phi$ $ds_\phi = \hat{\boldsymbol{\phi}} R dR d\theta$
Differential volume, $dV =$	$dx dy dz$	$r dr d\phi dz$	$R^2 \sin\theta dR d\theta d\phi$

GRADIENT, DIVERGENCE, CURL, & LAPLACIAN OPERATORS

CARTESIAN (RECTANGULAR) COORDINATES (x, y, z)

$$\nabla V = \mathbf{x} \frac{\partial V}{\partial x} + \mathbf{y} \frac{\partial V}{\partial y} + \mathbf{z} \frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \mathbf{x} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{y} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{z} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

CYLINDRICAL COORDINATES (r, ϕ, z)

$$\nabla V = \mathbf{r} \frac{\partial V}{\partial r} + \phi \frac{1}{r} \frac{\partial V}{\partial \phi} + \mathbf{z} \frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} (rA_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r} \begin{vmatrix} \mathbf{r} & \phi \mathbf{r} & \mathbf{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \mathbf{r} \left(\frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) + \phi \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \mathbf{z} \frac{1}{r} \left[\frac{\partial}{\partial r} (rA_\phi) - \frac{\partial A_r}{\partial \phi} \right]$$

$$\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

SPHERICAL COORDINATES (R, θ, ϕ)

$$\nabla V = \mathbf{R} \frac{\partial V}{\partial R} + \theta \frac{1}{R} \frac{\partial V}{\partial \theta} + \phi \frac{1}{R \sin \theta} \frac{\partial V}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 A_R) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{R \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2 \sin \theta} \begin{vmatrix} \mathbf{R} & \theta \mathbf{R} & \phi R \sin \theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & R A_\theta & (R \sin \theta) A_\phi \end{vmatrix}$$

$$= \mathbf{R} \frac{1}{R \sin \theta} \left[\frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{\partial A_\theta}{\partial \phi} \right] + \theta \frac{1}{R} \left[\frac{1}{\sin \theta} \frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R} (R A_\phi) \right] + \phi \frac{1}{R} \left[\frac{\partial}{\partial R} (R A_\theta) - \frac{\partial A_R}{\partial \theta} \right]$$

$$\nabla^2 V = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial V}{\partial R} \right) + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2}$$