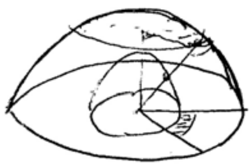
1. (30%) A section of a sphere is described by $1 \le R \le 5$, $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le 2\pi$. A vector field $\vec{E} = \frac{1}{4\pi R^2}\hat{R} + \hat{\theta}\frac{\sin\theta}{R}$. Verify divergence theorem by calculating: $\oiint_s \vec{E}\cdot d\vec{s} = \iiint_v (\nabla\cdot\vec{E})dv$.

$$\oiint \vec{E}\cdot d\vec{s} = \iint_{\text{Inner}} \vec{E}\cdot d\vec{s} + \iint_{\text{outter}} \vec{E}\cdot d\vec{s}$$



$$+ \iint_{\text{bottom}} \vec{E}\cdot d\vec{s}$$

$$\iint_{\text{inner}} \vec{E}\cdot d\vec{s} = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\left(\frac{1}{4\pi R^2}\hat{R} + \hat{\theta}\frac{\sin\theta}{R}\right)(-\hat{R})\, R\,d\theta\cdot R\sin\theta\,d\phi$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta\,d\theta \cdot \int_0^{2\pi} d\phi \;\; \frac{-1}{4\pi}\sin\theta \Big|_{R=1}$$

$$= -2\pi\cdot\frac{1}{4\pi}\cdot 1 = -\frac{1}{2}$$

$$\iint_{\text{Outter}} \vec{E}\cdot d\vec{s} = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\left(\frac{1}{4\pi R^2}\hat{R} + \hat{\theta}\frac{\sin\theta}{R}\right)\cdot R^2\sin\theta\,d\ldots\Big|_{R=5}$$

$$= \frac{1}{2}$$

$$\iint_{\text{bottom}} \vec{E}\cdot d\vec{s} = \int_0^{2\pi} d\phi \int_1^5 \left(\frac{1}{4\pi R^2}\hat{R} + \hat{\theta}\frac{\sin\theta}{R}\right)\hat{\theta}\; dR \,\Big|_{\theta=}\; R\sin\ldots$$

$$= \int_0^{2\pi} d\phi \int_1^5 \sin^2\theta \cdot dR \,\Big|_{\theta=\frac{\pi}{2}}$$

$$= 2\pi\cdot 4 = 8\pi$$

$$\nabla\cdot\vec{E} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\cdot\frac{1}{4\pi R^2}\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta}{R}\right)$$

$$= 0 + \frac{1}{R^2\sin\theta}\cdot 2\sin\theta\cos\theta$$

$$= 0 + \frac{1}{\ldots}2\cos\theta$$

$$\iiint (\nabla \cdot \vec{E})\, dv = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_1^5 \frac{2}{\cancel{R^2}} \cos\theta \cdot \cancel{R^2} \sin\theta \, d\theta \, d\phi \, d$$

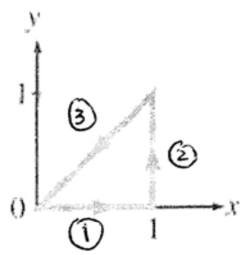
$$= \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} -2\cos\theta \, d\cos\theta \cdot \int_1^5 dR$$

$$= 2\pi \cdot \left(-2\frac{\cos^2\theta}{2}\right)\Big|_0^{\frac{\pi}{2}} \cdot 4$$

$$= 8\pi$$

$$\oiint \vec{E} \cdot d\vec{S} = \iiint (\nabla \cdot \vec{E})\, dv \quad \checkmark$$

2. (35%) Verify Stokes's theorem for the vector field $\vec{A} = \hat{x}y^3 + \hat{y}x^3$ along the contour shown below:



$$\oint \vec{A}\cdot d\vec{l} = \iint (\nabla \times \vec{A})\cdot d\vec{s}$$

$$\oint \vec{A}\cdot d\vec{l} = \int_{①} \vec{A}\cdot d\vec{l} + \int_{②} \vec{A}\cdot d\vec{l} + \int_{③} \vec{A}\cdot d\vec{l}$$

$$\int_{①} \vec{A}\cdot d\vec{l} = \int_0^1 (\hat{x}y^3 + \hat{y}x^3)\cdot \hat{x}\,dx \Big|_{y=}$$

$$= \int_0^1 y^3\,dx \Big|_{y=0} = 0$$

$$\int_{②} \vec{A}\cdot d\vec{l} = \int_0^1 (\hat{x}y^3 + \hat{y}x^3)\cdot \hat{y}\,dy \Big|_{x=1} = \int_0^1 x^3\,dy \Big|_{\cancel{x}=1}$$

$$= \cancel{\tfrac{1}{2}}\ 1$$

$$\int_{③} \vec{A}\cdot d\vec{l} = \int_0^1 (\hat{x}y^3 + \hat{y}x^3)(-\hat{x}dx - \hat{y}dy)$$

$$= \int_0^1 -y^3dx - x^3dy \Big|_{x=y}.$$

$$= \int_0^1 -x^3\,dx + \int_0^1 -y\,dy = -2\int_0^1 x^3\,dx$$

$$= -2\cdot \frac{x^4}{4}\Big|_0^1 = \cancel{\phi} -\frac{1}{2}$$

$$\therefore \oint \vec{A}\cdot d\vec{l} = 0 + \cancel{1} - \frac{1}{2} = \cancel{0} + \frac{1}{2}$$

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 & x^3 & 0 \end{vmatrix} = \cancel{\hat{x}(2x^2-2y^2)+\hat{y}\,0+}$$

$$= \hat{x}\,0 + \hat{y}\,0 + \hat{z}(2x^2-2y^2$$

$$\int_0^1\int_0^x \hat{z}\,(3x^2-3y^2)\,dx\,dy$$

$$= \int_0^1 dx \int_0^x (3x^2-3y^2)\,dy$$

$$= \int_0^1 dx \int_0^x 3x^2\,dy \;-\; \int_0^1 dx \int_0^x +3y^2\,dy$$

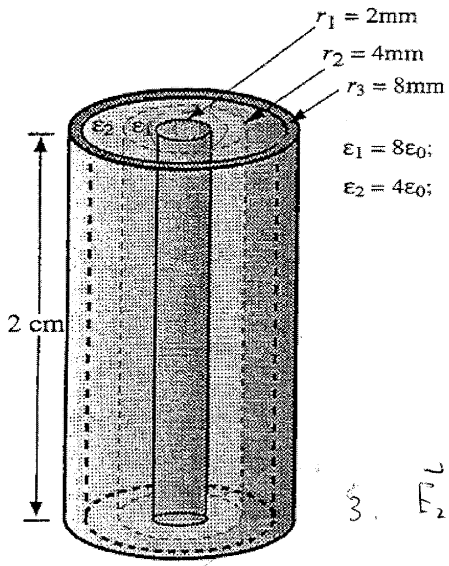
$$= \int_0^1 3x^3\,dx \;-\; \int_0^1 \frac{3}{3}x^3\,dx$$

$$\cancel{= \int_0^1 \frac{4}{3}x^3\,dx = \frac{4}{3}\cdot\frac{x^4}{4}\Big|_0^1 = \frac{1}{3}}$$

$$= \int_0^1 2x^3\,dx = 2\cdot\frac{1}{4} = \frac{1}{2}$$

$$\oint \vec{A}\cdot d\vec{l} = \iint \vec{A}\cdot d\vec{s}$$

3. (35 points) A coaxial capacitor with inner connector radius $r_1 = 2mm$, and outer connector radius $r_3 = 8mm$, is filled with two different materials as shown in the following figure. The length of the capacitor is 2cm. Calculate the capacitance of the capacitor.



1. assuming surface charge density σ

2. $\vec{E}_1 = \hat{r}\,\frac{2\pi r_1 \sigma}{2\pi r \varepsilon_1} = \hat{r}\,\frac{r_1\sigma}{r\varepsilon_1}$

$V_1 = \int_{r_1}^{r_2} \vec{E}_1 \cdot d\vec{r}$

$= \int_{r_1}^{r_2} \frac{r_1\sigma}{r\varepsilon_1}\, dr = \frac{r_1\sigma}{\varepsilon_1}\ln\frac{r_2}{r_2}$

3. $\vec{E}_2 = \hat{r}\,\frac{r_1\sigma}{r\varepsilon_2}$

$V_2 = \int_{r_2}^{r_2} \vec{E}_2 \cdot d\vec{r} = \frac{r_1\sigma}{\varepsilon_2}\ln\frac{r_3}{r_2}$

4. $V_{total} = V_1 + V_2 = \frac{r_1\sigma}{\varepsilon_1}\ln\frac{r_2}{r_1} + \frac{r_1\sigma}{\varepsilon_2}\ln\frac{r_3}{r_2}$

$Q = 2\pi r_1 \sigma$

$\Rightarrow C = \frac{Q}{V} = \frac{2\pi r_1}{\frac{r_1}{\varepsilon_1}\ln\frac{r_2}{r_1} + \frac{r_1}{\varepsilon_2}\ln\frac{r_3}{r_2}}$

$= \frac{2\pi\, 2\times10^{-3}}{\frac{2\times10^{-3}}{8\varepsilon_0}\ln 2 + \frac{2\times10^{-3}}{4\varepsilon_0}\ln 2} = \frac{2\pi\varepsilon_0}{\left(\frac{1}{8}+\frac{1}{4}\right)\ln 2} = \frac{2\pi\varepsilon_0}{\frac{3}{8}\ln 2}$

$= \frac{16\pi\varepsilon_0}{3\ln 2}$