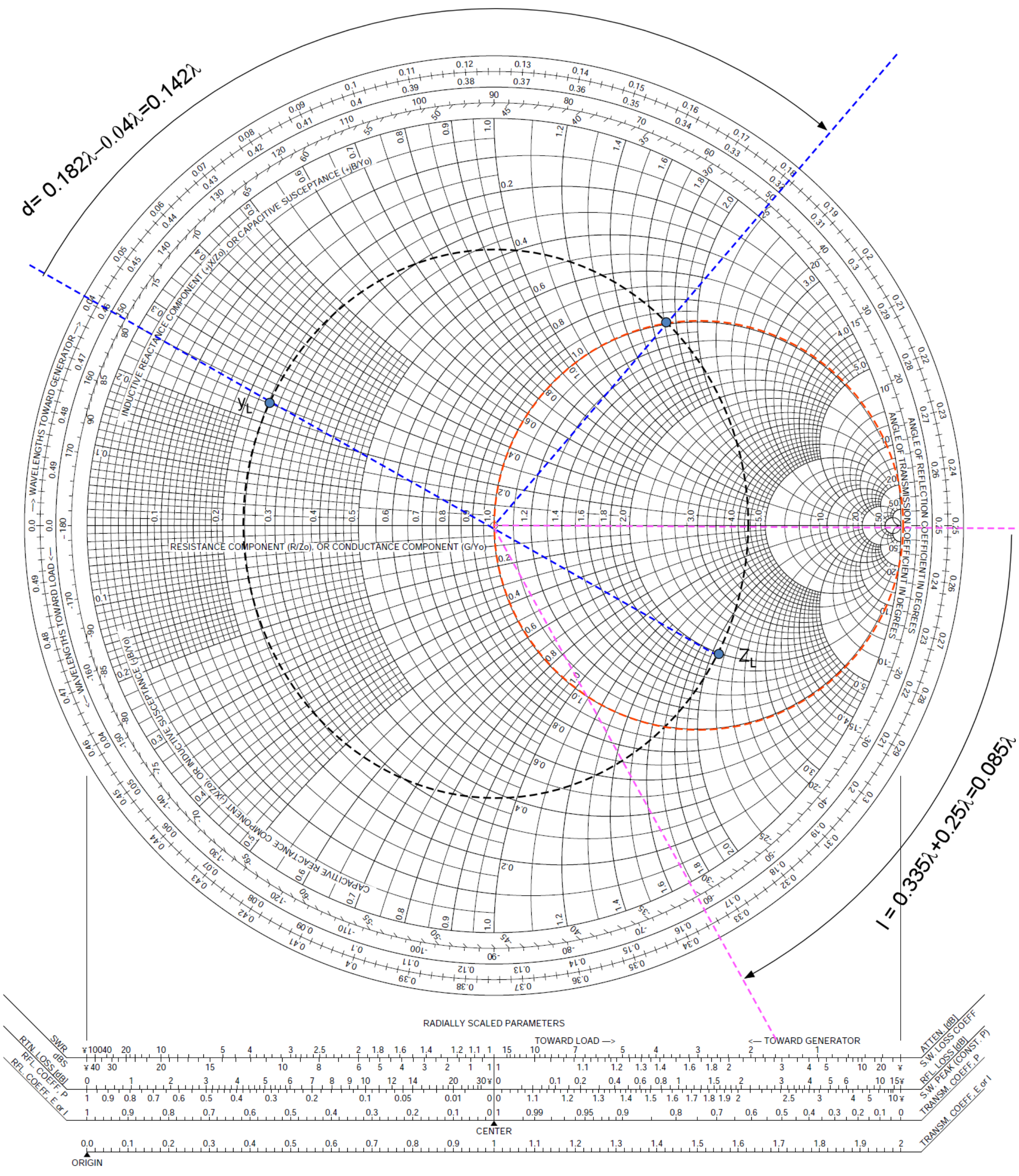
EECE.3600 Exam II

04/06/2018

Name: ______________________________

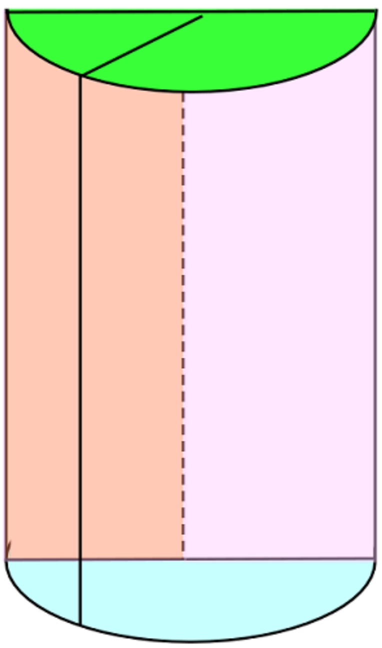
Signature: ______________________________

1. (35%) On a lossless 50-Ω transmission line terminated with a $Z_L = 100\text{-}j100\ \Omega$. If this transmission line is to be matched to the load using a shorted load stub. Determine the stub length and distance between the load and stub. Two possible answers. You only need to show one of them.



2. (30%) For a vector filed $\vec{A} = r^2\hat{r} + 3r\phi\hat{\phi} - 2\hat{z}$, verify the divergence theorem

$\oint\!\!\oint_s \vec{A}\cdot d\vec{s} = \iiint_v (\nabla\cdot\vec{A})dv$, on a section of a cylinder bounded by $r = 1,\ -\frac{\pi}{2} \le \phi \le \frac{\pi}{2},\ 1 \le z \le 3.$



Solution:

$$\oint\!\!\oint_s \vec{A}\cdot d\vec{s} = \iint_{top} \vec{A}\cdot d\vec{s} + \iint_{bottom} \vec{A}\cdot d\vec{s} + \iint_{side} \vec{A}\cdot d\vec{s} + \iint_{left} \vec{A}\cdot d\vec{s} + \iint_{right} \vec{A}\cdot d\vec{s},$$

$$\iint_{top} \vec{A}\cdot d\vec{s} = \iint_{top} -2\hat{z}\cdot\hat{z}rdrd\phi = -\pi, \quad \iint_{bottom} \vec{A}\cdot d\vec{s} = \iint_{bottom} -2\hat{z}\cdot(-\hat{z})rdrd\phi = \pi,$$

$$\iint_{side} \vec{A}\cdot d\vec{s} = \iint_{side} r^2\hat{r}\cdot\hat{r}rd\phi dz\Big|_{r=1} = 2\pi,$$

$$\iint_{left} \vec{A}\cdot d\vec{s} = \iint_{left} 3r\phi\hat{\phi}\cdot\hat{\phi}drdz\Big|_{\phi=\pi/2} = \frac{3\pi}{2},$$

$$\iint_{right} \vec{A}\cdot d\vec{s} = \iint_{right} 3r\phi\hat{\phi}\cdot\left(-\hat{\phi}\right)drdz\Big|_{\phi=-\pi/2} = \frac{3\pi}{2},$$

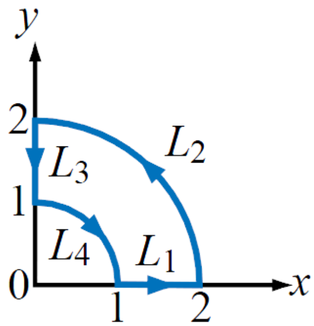
$$\oint\!\!\oint_s \vec{A}\cdot d\vec{s} = \iint_{top} \vec{A}\cdot d\vec{s} + \iint_{bottom} \vec{A}\cdot d\vec{s} + \iint_{side} \vec{A}\cdot d\vec{s} + \iint_{left} \vec{A}\cdot d\vec{s} + \iint_{right} \vec{A}\cdot d\vec{s} = 5\pi.$$

$$\nabla\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}\left(rr^2\right) + \frac{1}{r}\frac{\partial}{\partial\phi}\left(3r\phi\right) = 3r + 3,$$

$$\iiint_v (\nabla\cdot\vec{A})dv = \int_1^3 dz\int_{-\pi/2}^{-\pi/2} d\phi\int_0^1 (3r+3)rdr = 2\pi\left(r^3 + \frac{3}{2}r^2\right)\Big|_0^1 = 5\pi,$$

$$\oint\!\!\oint_s \vec{A}\cdot d\vec{s} = \iiint_v (\nabla\cdot\vec{A})dv.$$

3. (35%) Verify Stokes's theorem for the vector field $\vec{A} = \hat{r} r cos\phi + \hat{\phi} sin\phi$ along the contour shown below:



Solution:

$$\oint \mathbf{B}\cdot d\mathbf{l} = \int_{L_1} \mathbf{B}\cdot d\mathbf{l} + \int_{L_2} \mathbf{B}\cdot d\mathbf{l} + \int_{L_3} \mathbf{B}\cdot d\mathbf{l} + \int_{L_4} \mathbf{B}\cdot d\mathbf{l},$$

$$\mathbf{B}\cdot d\mathbf{l} = (\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)\cdot(\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}} r\,d\phi + \hat{\mathbf{z}}\,dz) = r\cos\phi\,dr + r\sin\phi\,d\phi,$$

$$\int_{L_1} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=1}^{2} r\cos\phi\,dr\right)\Big|_{\phi=0,\ z=0} + \left(\int_{\phi=0}^{0} r\sin\phi\,d\phi\right)\Big|_{z=0}$$

$$= \left(\tfrac{1}{2}r^2\right)\Big|_{r=1}^{2} + 0 = \frac{3}{2},$$

$$\int_{L_2} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{2} r\cos\phi\,dr\right)\Big|_{z=0} + \left(\int_{\phi=0}^{\pi/} r\sin\phi\,d\phi\right)\Big|_{r=2,\ z=0}$$

$$= 0 + (-2\cos\phi)\Big|_{\phi=0}^{\pi/2} = 2,$$

$$\int_{L_3} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{1} r\cos\phi\,dr\right)\Big|_{\phi=\pi/2,\ z=0} + \left(\int_{\phi=\pi/2}^{\pi/2} r\sin\phi\,d\phi\right)\Big|_{z=0} = 0,$$

$$\int_{L_4} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=1}^{1} r\cos\phi\,dr\right)\Big|_{z=0} + \left(\int_{\phi=\pi/2}^{0} r\sin\phi\,d\phi\right)\Big|_{r=1,\ z=0}$$

$$= 0 + (-\cos\phi)\Big|_{\phi=\pi/2}^{0} = -1,$$

$$\oint \mathbf{B}\cdot d\mathbf{l} = \frac{3}{2} + 2 + 0 - 1 = \frac{5}{2}.$$

$$\nabla\times\mathbf{B} = \nabla\times(\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)$$

$$= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial}{\partial\phi}0 - \frac{\partial}{\partial z}(\sin\phi)\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial}{\partial z}(r\cos\phi) - \frac{\partial}{\partial r}0\right)$$

$$+ \hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}(r(\sin\phi)) - \frac{\partial}{\partial\phi}(r\cos\phi)\right)$$

$$= \hat{\mathbf{r}}0 + \hat{\boldsymbol{\phi}}0 + \hat{\mathbf{z}}\frac{1}{r}(\sin\phi + (r\sin\phi)) = \hat{\mathbf{z}}\sin\phi\left(1 + \frac{1}{r}\right),$$

$$\iint \nabla\times\mathbf{B}\cdot ds = \int_{\phi=0}^{\pi/2}\int_{r=1}^{2}\left(\hat{\mathbf{z}}\sin\phi\left(1+\frac{1}{r}\right)\right)\cdot(\hat{\mathbf{z}} r\,dr\,d\phi)$$

$$= \int_{\phi=0}^{\pi/2}\int_{r=1}^{2} \sin\phi\,(r+1)\,dr\,d\phi$$

$$= \left(\left(-\cos\phi\left(\tfrac{1}{2}r^2 + r\right)\right)\Big|_{r=1}^{2}\right)\Big|_{\phi=0}^{\pi/2} = \frac{5}{2}.$$

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation, A =	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A, $\|A\| =$	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}} \cdot \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product, A · B =	$A_x B_x + A_y B_y + A_z B_z$	$A_r B_r + A_\phi B_\phi + A_z B_z$	$A_R B_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product, A × B =	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length, $d\mathbf{l}$ =	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume, $d\mathcal{V}$ =	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

GRADIENT, DIVERGENCE, CURL, & LAPLACIAN OPERATORS

CARTESIAN (RECTANGULAR) COORDINATES (x, y, z)

$$\nabla V = \mathbf{x}\frac{\partial V}{\partial x} + \mathbf{y}\frac{\partial V}{\partial y} + \mathbf{z}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \mathbf{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

CYLINDRICAL COORDINATES (r, ϕ, z)

$$\nabla V = \mathbf{r}\frac{\partial V}{\partial r} + \boldsymbol{\phi}\frac{1}{r}\frac{\partial V}{\partial \phi} + \mathbf{z}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \mathbf{r} & \boldsymbol{\phi} r & \mathbf{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \mathbf{r}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \boldsymbol{\phi}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \mathbf{z}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

SPHERICAL COORDINATES (R, θ, ϕ)

$$\nabla V = \mathbf{R}\frac{\partial V}{\partial R} + \boldsymbol{\theta}\frac{1}{R}\frac{\partial V}{\partial \theta} + \boldsymbol{\phi}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \mathbf{R} & \boldsymbol{\theta} R & \boldsymbol{\phi} R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix}$$

$$= \mathbf{R}\frac{1}{R\sin\theta}\left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right] + \boldsymbol{\theta}\frac{1}{R}\left[\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R}(RA_\phi)\right] + \boldsymbol{\phi}\frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial \theta}\right]$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$