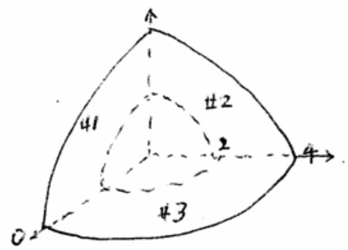
1. (30 points) A section of a sphere is described by $2 \le R \le 4$, $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le \frac{\pi}{2}$. A vector field $\vec{E} = \frac{1}{4\pi R^2}\hat{R}$. Verify divergence theorem by calculating: $\oiint_s \vec{E} \cdot d\vec{s} = \iiint_v (\nabla \cdot \vec{E}) dv$.



$$\nabla \cdot \vec{E} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 E_R)$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \cdot \frac{1}{4\pi R^2}\right) = 0$$

$$\Rightarrow \iiint (\nabla \cdot \vec{E})\, dV = 0$$

$$\oiint \vec{E} \cdot d\vec{s} = \iint_{\text{outter}} \vec{E} \cdot d\vec{s} + \iint_{\text{inner}} \vec{E} \cdot d\vec{s} + \iint_{\#1} \vec{E} \cdot d\vec{s} + \iint_{\#2} \vec{E} \cdot d\vec{s} + \iint_{\#3} \vec{E}$$

$$\iint_{\text{outter}} \vec{E} \cdot d\vec{s} = \iint \hat{R}\frac{1}{4\pi R^2} \cdot \hat{R}\; R^2 \sin\phi \, d\theta \, d\phi \Big|_{R=4}$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{1}{4\pi} \cdot \sin\phi \, d\theta \, d\phi = \frac{1}{4\pi} \cdot \frac{\pi}{2} = \frac{1}{8}$$

$$\iint_{\text{inner}} \vec{E} \cdot d\vec{s} = \iint \hat{R}\frac{1}{4\pi R^2} \cdot (-\hat{R})\, R^2 \sin\phi \, d\theta \, d\phi \Big|_{R=2} = -\frac{1}{8}$$

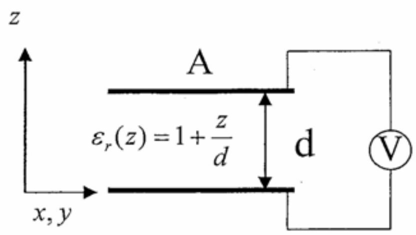
$$\iint_{\#1} \vec{E} \cdot d\vec{s} = \int_0^{\frac{\pi}{2}}\int_2^4 \hat{R}\frac{1}{4\pi R^2} \cdot (-\hat{\phi})\, R \, d\theta \, dR \Big|_{\phi=0} = 0$$

$$\iint_{\#2} \vec{E} \cdot d\vec{s} = \int_0^{\frac{\pi}{2}}\int_2^4 \hat{R}\frac{1}{4\pi R^2} \cdot (\hat{\phi})\, R \, d\theta \, dR \Big|_{\phi=\frac{\pi}{2}} = 0$$

$$\iint_{\#3} \vec{E} \cdot d\vec{s} = \int\int \hat{R}\frac{1}{4\pi R^2} \cdot (\hat{\theta})\, R \sin\theta \, dR \, d\phi \Big|_{\theta=\frac{\pi}{2}} = 0$$

$$\Rightarrow \oiint \vec{E} \cdot d\vec{s} = \iiint (\nabla \cdot \vec{E})\, dV$$

2. (30 points) A parallel-plate capacitor has the dielectric filling region between the plate. The relative permittivity is $\varepsilon_r(z) = 1 + z/d$, and A1 = 1cm^2, d = 2cm, V=20V, ignore Fringing field effect.



Determine:

(a) The Electric field ~~density~~ E of the region.

(b) The capacitance of the capacitor

Solution: (a) $\vec{E} = (-\hat{z})\dfrac{Q/A}{\varepsilon_r \varepsilon_0}$, assuming the charge

$$20\,(V) = V_{drop} = \int \vec{E} \cdot d\vec{l} = \int_0^d \frac{Q/A}{\varepsilon_r \varepsilon_0} \cdot dz$$

$$= \int_0^d \frac{Q/A}{(1+\frac{z}{d})\,\varepsilon_0}\, dz$$

$$= \frac{Q}{A\varepsilon_0} \cdot d \ln\left(1+\frac{z}{d}\right)\Big|_0^d$$

$$= \frac{Q}{A\varepsilon_0}\, d \ln 2$$

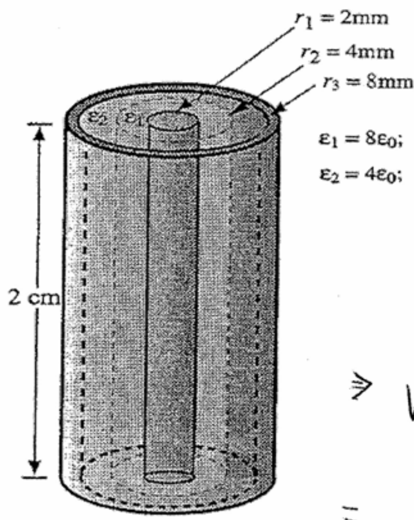
$$\Rightarrow \frac{Q}{A\varepsilon_0} = \frac{20}{d \ln 2}$$

$$\Rightarrow \vec{E} = -\hat{z}\,\frac{20}{d\ln 2}\,\frac{1}{(1+\frac{z}{d})}$$

(b) $$C \equiv \frac{Q}{V} = \frac{A\varepsilon_0}{d \ln 2} = \frac{1\times 10^{-4} \times 8.85\times}{\ln 2}$$

$$= [illegible] \times 10^{-13}\ (F)$$

3. (30 points) A coaxial capacitor with inner connector radius $r_1 = 2mm$, and outer connector radius $r_2 = 8mm$, is filled with two different materials as shown in the following figure. The length of the capacitor is 2cm. Calculate the capacitance of the capacitor.



Solution:

assuming the charge carried is Q

$$\vec{E}_1 = \frac{Q}{2\pi L \varepsilon_1 r} \quad \text{for region } \underline{1}$$

$$\vec{E}_2 = \frac{Q}{2\pi L \varepsilon_2 r}, \quad \text{for region}$$

$$\Rightarrow V_{drop} = \int_{r_1}^{r_2} \vec{E}_1 \cdot d\vec{l} + \int_{r_2}^{r_3} \vec{E}_2 \cdot d\vec{l}$$

$$= \frac{Q}{2\pi L \varepsilon_1} \ln \frac{r_2}{r_1} + \frac{Q}{2\pi L \varepsilon_2} \ln \frac{r_3}{r_2}$$

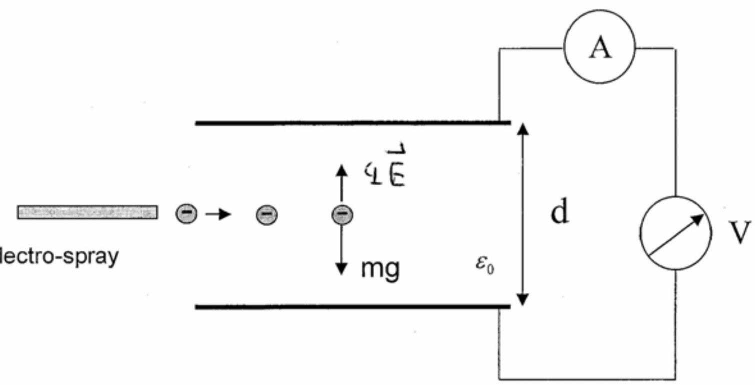
$$\Rightarrow \frac{1}{C} = \frac{1}{2\pi L \varepsilon_1} \ln \frac{r_2}{r_1} + \frac{1}{2\pi L \varepsilon_2} \ln \frac{r_3}{r_2}$$

$$= \frac{1}{2\pi \, 2\times10^{-2} \cdot 8.85\times10^{-12}} \left(\frac{1}{8} \ln 2 + \frac{1}{4} \ln 2 \right)$$

$$\Rightarrow C = \cancel{43\,(pF)}$$

4.3 pF

4. An electro-spray (shown in the following figure) can be used to determine the charge of a particle has. Assuming the each particle has the same weight, i.e. assuming the gravity force is 1×10^{-12} N for each particle. The particles are negatively charged. If the gravity force is balanced by the electric field, the charge won't fall on the parallel plates. In this case, the current measured in the circuit would be zero. Otherwise, there current won't be zero. We know that d is 1mm and we observed that the current is zero when V = 62.5(V).



(a) (10 points) If each particle has the same amount of charge, what's the charge for each particle? How many electrons it has?

Solution:

$$\vec{E} = \frac{V}{d} = \frac{62.5\ (V)}{1mm} = 6.25\times10^{4}\ (V/m)$$

$$q\vec{E} = 1\times10^{-12}\ N$$

$$\Rightarrow \quad q = \frac{1\times10^{-12}\ (N)}{6.25\times10^{4}\ (V/m)} = 1.6\times10^{-17}\ (C)$$

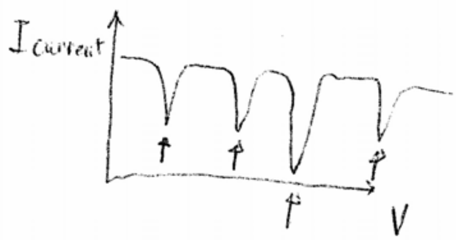
$$\#\text{ of electrons } N = \frac{q}{e} = \frac{1.6\times10^{-17}\ (C)}{1.6\times10^{-19}\ (C)}$$

$$= 100$$

(b) (10 bonus points) if the particles have different charges, can you measure the charge distribution of the particles? How? What are the resolutions the voltage and the current meters need to have in order to resolve the charge difference by a single electron?

Solution: the Charge distribution can be measure by monitoring the ~~V~~ Imin s. I.e.

tuning the voltage, particles that have the same Charge wound suspend there in this case you will see current has a dip.



suppose gravity force is still 10^{-12} (N).

To monitor a particle with 101 electrons

$q\vec{E} = 1\times10^{-12}$, then $\vec{E} = \dfrac{1\times10^{-12}}{101\times1.6\times10^{-19}}$

then $V = \vec{E}\cdot d = \dfrac{1\times10^{-12}}{101\times1.6\times10^{-19}}\cdot 1$ ohm

$= 61.9$ (V)

So the resolution of V meter need to be larger than 0.5 (V)

For current. Suppose particles with 100 electrons, then $I = \dfrac{100 \text{ electron}}{\text{time}}$

$\sim 5.1\times10^{-22}$ A