1. (30 points) A section of a sphere is described by \( 2 \leq R \leq 4, \ 0 \leq \theta \leq \frac{\pi}{2}, \ 0 \leq \phi \leq \frac{\pi}{2} \). A vector field \( \vec{E} = \frac{1}{4\pi R^2} \hat{R} \). Verify divergence theorem by calculating: \( \iiint_{V} (\nabla \cdot \vec{E}) \, dV = 0 \).

\[
\nabla \cdot \vec{E} = \frac{1}{R^2} \frac{3}{\theta R} \left( R^2 E_R \right) = 0
\]

\[
\Rightarrow \iiint_{V} (\nabla \cdot \vec{E}) \, dV = 0
\]

\[
\iiint_{V} \vec{E} \cdot d\vec{S} = \int_{\text{outer}} \vec{E} \cdot d\vec{S} + \int_{\text{inner}} \vec{E} \cdot d\vec{S} + \int_{\#1} \vec{E} \cdot d\vec{S} + \int_{\#2} \vec{E} \cdot d\vec{S} + \int_{\#3} \vec{E} \cdot d\vec{S}
\]

\[
\int_{\text{outer}} \vec{E} \cdot d\vec{S} = \int_{R=0}^{R=4} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\phi=0}^{\phi=\frac{\pi}{2}} \frac{1}{4\pi R^2} \hat{R} \cdot \hat{R} R^2 \sin \phi \, d\theta \, d\phi = \frac{1}{4\pi} \cdot \frac{64}{2} = \frac{1}{8}
\]

\[
\int_{\text{inner}} \vec{E} \cdot d\vec{S} = \int_{R=2}^{R=4} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\phi=0}^{\phi=\frac{\pi}{2}} \frac{1}{4\pi R^2} \hat{R} \cdot (-\hat{\phi}) R \sin \phi \, d\theta \, d\phi = -\frac{1}{8}
\]

\[
\int_{\#1} \vec{E} \cdot d\vec{S} = \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{R=2}^{R=4} \frac{1}{4\pi R^2} \hat{R} \cdot (-\hat{\phi}) R \, dR \, d\phi = 0
\]

\[
\int_{\#2} \vec{E} \cdot d\vec{S} = \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{R=2}^{R=4} \frac{1}{4\pi R^2} \hat{R} \cdot (-\hat{\phi}) R \, dR \, d\phi = 0
\]

\[
\int_{\#3} \vec{E} \cdot d\vec{S} = \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{R=2}^{R=4} \frac{1}{4\pi R^2} \hat{R} \cdot (-\hat{\phi}) R \sin \phi \, dR \, d\phi = 0
\]

\[
\Rightarrow \iiint_{V} \vec{E} \cdot d\vec{S} = \iiint_{V} (\nabla \cdot \vec{E}) \, dV
\]
2. (30 points) A parallel-plate capacitor has the dielectric filling region between the plate. The relative permittivity is $\epsilon_r(z) = 1 + \frac{z}{d}$, and $A = 1 \text{cm}^2$, $d = 2 \text{cm}$, $V = 20 \text{V}$, ignore Fringing field effect.

![Diagram of a parallel-plate capacitor with dielectric](image)

Determine:
(a) The Electric field density $\vec{E}$ of the region.
(b) The capacitance of the capacitor

**Solution: (a)**

$$\vec{E} = \left(\frac{Q/A}{\epsilon_r \epsilon_0}\right)$$

Assuming the charge is uniform across the plate,

$$2 \pi \psi = V = \int_{\text{drop}} \vec{E} \cdot d\vec{l} = \int_0^d \frac{Q/A}{\epsilon_r \epsilon_0} \, dz$$

$$= \int_0^d \frac{Q/A}{(1 + \frac{z}{d}) \epsilon_0} \, dz$$

$$= \frac{Q}{A \epsilon_0} \left[ d \ln \left(1 + \frac{z}{d}\right) \right]_0^d$$

$$= \frac{Q}{A \epsilon_0} \left[ d \ln \left(1 + \frac{2}{2}\right) \right]_0$$

$$= \frac{Q}{A \epsilon_0} \cdot d \ln 2$$

$$\Rightarrow \quad \frac{Q}{A \epsilon_0} = \frac{2 \psi}{d \ln 2}$$

$$\Rightarrow \quad E = -\frac{2 \psi}{d \ln 2} \left(1 + \frac{z}{d}\right)$$

**Solution: (b)**

$$C = \frac{Q}{V} = \frac{A \epsilon_0}{d \ln 2} = \frac{1 \times 10^{-6} \times 8.85 \times 10^{-12}}{\ln 2}$$

$$= 13 \times 10^{-12} \text{ F}$$
3. (30 points) A coaxial capacitor with inner connector radius $r_1 = 2\text{mm}$, and outer connector radius $r_2 = 8\text{mm}$, is filled with two different materials as shown in the following figure. The length of the capacitor is 2cm. Calculate the capacitance of the capacitor.

\[ r_1 = 2\text{mm} \]
\[ r_2 = 4\text{mm} \]
\[ r_3 = 8\text{mm} \]
\[ \varepsilon_1 = 8\varepsilon_0 \]
\[ \varepsilon_2 = 4\varepsilon_0 \]

\[ 2 \text{cm} \]

**Solution:**

Assuming the charge is $Q$

\[ E_1 = \frac{Q}{2\pi \varepsilon_1 r_1} \text{ for region 1} \]
\[ E_2 = \frac{Q}{2\pi \varepsilon_2 r_2} \text{ for region 2} \]

\[ V_{\text{drop}} = \int_{r_1}^{r_2} E_1 \cdot dl + \int_{r_2}^{r_3} E_2 \cdot dl \]

\[ = \frac{Q}{2\pi \varepsilon_1} \left( \ln \frac{r_2}{r_1} \right) + \frac{Q}{2\pi \varepsilon_2} \left( \ln \frac{r_3}{r_2} \right) \]

\[ \Rightarrow \]

\[ \frac{1}{C} = \frac{1}{2\pi \varepsilon_1} \left( \ln \frac{r_2}{r_1} \right) + \frac{1}{2\pi \varepsilon_2} \left( \ln \frac{r_3}{r_2} \right) \]

\[ = \frac{1}{2\pi \varepsilon_0 \cdot 8.85 \times 10^{-12}} \left( \frac{1}{8} \ln 2 + \frac{1}{4} \ln 2 \right) \]

\[ \Rightarrow \]

\[ C = 4.3 \ \mu\text{F} \]
4. An electro-spray (shown in the following figure) can be used to determine the charge of a particle has. Assuming the each particle has the same weight, i.e. assuming the gravity force is $1 \times 10^{-12}$ N for each particle. The particles are negatively charged. If the gravity force is balanced by the electric field, the charge won’t fall on the parallel plates. In this case, the current measured in the circuit would be zero. Otherwise, there current won’t be zero. We know that $d$ is 1 mm and we observed that the current is zero when $V = 62.5$ (V).

(a) (10 points) If each particle has the same amount of charge, what’s the charge for each particle? How many electrons it has?

**Solution:**

\[
E = \frac{V}{d} = \frac{62.5 \text{ (V)}}{1 \text{ mm}} = 6.25 \times 10^4 \text{ (V/mm)}
\]

\[
q \frac{E}{d} = 1 \times 10^{-12} \text{ N}
\]

\[
\Rightarrow \quad q = \frac{1 \times 10^{-12} \text{ (N)}}{6.25 \times 10^4 \text{ (V/mm)}} = 1.6 \times 10^{-17} \text{ (C)}
\]

\[
\# \text{ of electrons} \quad N = \frac{q}{e} = \frac{1.6 \times 10^{-17} \text{ (C)}}{1.6 \times 10^{-19} \text{ (C)}} = 100
\]
(b) (10 bonus points) if the particles have different charges, can you measure the charge distribution of the particles? How? What are the resolutions the voltage and the current meters need to have in order to resolve the charge difference by a single electron?

Solution: the charge distribution can be measured by monitoring the current, i.e., tuning the voltage, particles that have the same charge would suspend there, in this case you will see current has a dip.

\[ I_{current} \]

Suppose gravity force is still \( 1 \times 10^{-12} \text{ cm} \).

To monitor a particle with 101 electrons

\[ qE = 1 \times 10^{-12}, \quad \text{then} \quad E = \frac{1 \times 10^{-12}}{101 \times 1.6 \times 10^{-19}} \]

then

\[ V = E \cdot d = \frac{1 \times 10^{-12}}{101 \times 1.6 \times 10^{-19}} \cdot 1 \text{ cm} \]

\[ = 61.9 \text{ V} \]

So the resolution of the voltage meters need to be larger than 0.5 V.

For current: Suppose 100 electrons, then

\[ I = \frac{100 \text{ electrons}}{\text{time}\times 1 \times 10^{-22} A} \]