EECE.3600 Exam II

11/14/2016

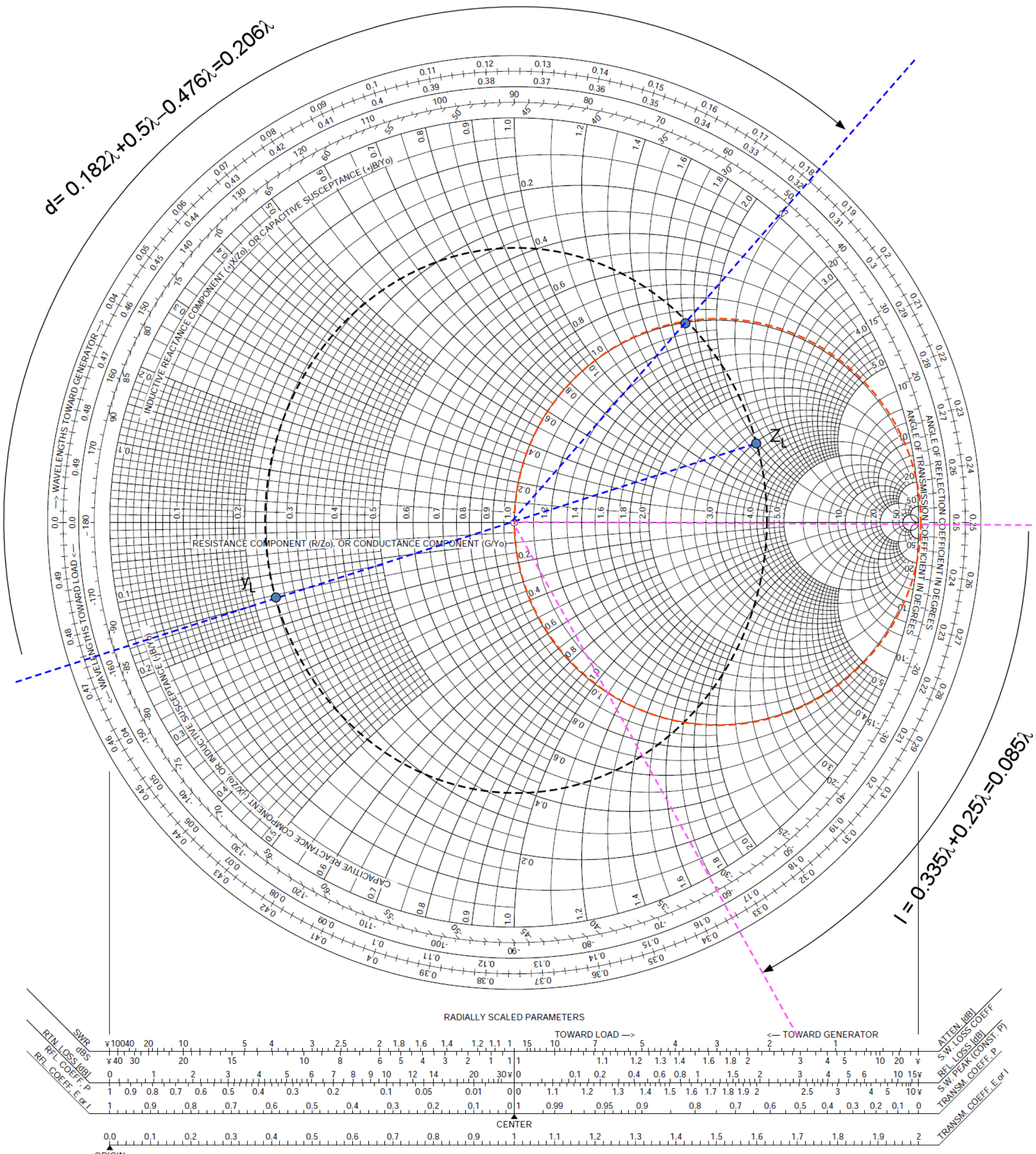
Name: ______________________________

Signature: ______________________________

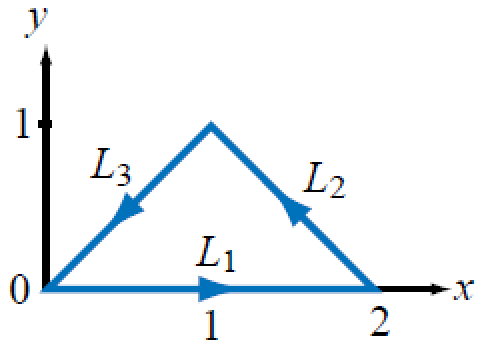
1. (35%) On a lossless 50-Ω transmission line terminated with a $Z_L = 150 + j100\ \Omega$. If this transmission line is matched to the load using a shorted load stub. Determine the stub length and distance between the load and stub. Two possible answers. You only need to show one of them.

Solution:

See Smith Chart



2. (30%) Verify Stokes's theorem for the vector field $\vec{A} = \hat{x}y^3 + \hat{y}x^3$ along the contour shown below:



Solution:

$$\oint \vec{A}\cdot d\vec{l} = \int_{L_1} \vec{A}\cdot d\vec{l} + \int_{L_2} \vec{A}\cdot d\vec{l} + \int_{L_3} \vec{A}\cdot d\vec{l}\ .$$

$$\int_{L_1} \vec{A}\cdot d\vec{l} = \int_{L_1} \left(\hat{x}y^3 + \hat{y}x^3\right)\cdot \hat{x}dx\,|_{y=0} = \int_{L_1} y^3 dx\,|_{y=0} = 0\,.$$

$$\int_{L_2} \vec{A}\cdot d\vec{l} = \int_2^1 \left(\hat{x}y^3 + \hat{y}x^3\right)\cdot \hat{x}dx\,|_{y=2-x} + \int_0^1 \left(\hat{x}y^3 + \hat{y}x^3\right)\cdot \hat{y}dy\,|_{x=2-y}$$

$$= \int_2^1 (2-x)^3 dx + \int_0^1 (2-y)^3 dy$$

$$= \int_1^2 (2-x)^3 d(2-x) - \int_0^1 (2-y)^3 d(2-y) \qquad .$$

$$= \frac{1}{4}(2-x)^4\Big|_1^2 - \frac{1}{4}(2-y)^4\,\Big|_0^1$$

$$= -\frac{1}{4} - \frac{1}{4}(1-16) = \frac{14}{4}$$

$$\int_{L_3} \vec{A}\cdot d\vec{l} = \int_1^0 \left(\hat{x}y^3 + \hat{y}x^3\right)\cdot \hat{x}dx\,|_{y=x} + \int_1^0 \left(\hat{x}y^3 + \hat{y}x^3\right)\cdot \hat{y}dy\,|_{x=y}$$

$$= \int_1^0 x^3 dx + \int_1^0 y^3 dy$$

$$= -\frac{1}{4}x^4\Big|_0^1 - \frac{1}{4}y^4\,\Big|_0^1 \qquad .$$

$$= -\frac{2}{4}$$

$$\oint \vec{A}\cdot d\vec{l} = \int_{L_1} \vec{A}\cdot d\vec{l} + \int_{L_2} \vec{A}\cdot d\vec{l} + \int_{L_3} \vec{A}\cdot d\vec{l} = 0 + \frac{14}{4} - \frac{2}{4} = 3\,.$$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^3 & x^3 & 0 \end{vmatrix} = \hat{x}0 + \hat{y}0 + \hat{z}\left(3x^2 - 3y^2\right)$$

$$\iint_s \left(\nabla \times \vec{A}\right) \cdot d\vec{s} = \iint_s \hat{z}\left(3x^2 - 3y^2\right) \cdot \hat{z}dxdy$$

$$= \int_0^1 dx \int_0^x \left(3x^2 - 3y^2\right)dy + \int_1^2 dx \int_0^{2-x} \left(3x^2 - 3y^2\right)dy$$

$$= \int_0^1 2x^3 dx + \int_1^2 \left[3x^2(2-x) - (2-x)^3\right]dx$$

$$= \frac{2}{4} + \int_1^2 3x^2(2-x)dx - \int_1^2 (2-x)^3 dx$$

$$= \frac{2}{4} + 2x^3\Big|_1^2 - \frac{3}{4}x^4\Big|_1^2 + \frac{1}{4}(2-x)^4\Big|_1^2$$

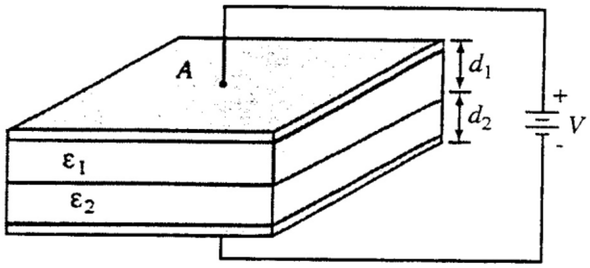
$$= \frac{2}{4} + 14 - \frac{3}{4}15 - \frac{1}{4}$$

$$= 3$$

$$\oint \vec{A} \cdot d\vec{l} = \int_s \left(\nabla \times \vec{A}\right) \cdot d\vec{s}$$

3. (35%) A Capacitor shown in the following figure consists of two dielectric layers. d1=d2 = 10cm, and A = 200cm^2. Determine the capacitance of the capacitor.

$\varepsilon_1 = 4\varepsilon_0 = 3.5\times10^{-11} F/m,\ \varepsilon_2 = 2\varepsilon_0 = 1.8\times10^{-11} F/m$.



Solution:

Assuming the surface charge density of σ, the flux density D is:

$\vec{D}_1 = \vec{D}_2 = -\hat{z}\sigma$.

$\vec{E}_1 = -\hat{z}\dfrac{\sigma}{\varepsilon_1}$, $\vec{E}_2 = -\hat{z}\dfrac{\sigma}{\varepsilon_2}$. $V = \vec{E}_1 d_1 + \vec{E}_2 d_2 = \dfrac{\sigma d_1}{\varepsilon_1} + \dfrac{\sigma d_2}{\varepsilon_2}$

$$C = \frac{Q}{V} = \frac{A\sigma}{\dfrac{\sigma d_1}{\varepsilon_1} + \dfrac{\sigma d_2}{\varepsilon_2}} = \frac{A}{\dfrac{d_1}{\varepsilon_1} + \dfrac{d_2}{\varepsilon_2}}$$

$$C = \frac{A}{\dfrac{d_1}{\varepsilon_1} + \dfrac{d_2}{\varepsilon_2}} = \frac{200\times10^{-4} m^2}{\dfrac{10\times10^{-2} m}{3.5\times10^{-11} F/m} + \dfrac{10\times10^{-2} m}{1.8\times10^{-11} F/m}} = \frac{2\times10^{-11} F}{\dfrac{10}{3.5} + \dfrac{10}{1.8}} = 2.4 pF .$$

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation, A =	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A, $\lvert A \rvert$ =	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1}$ =	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}} \cdot \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product, A · B =	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product, A × B =	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length, $d\mathbf{l}$ =	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume, $d\mathcal{V}$ =	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

GRADIENT, DIVERGENCE, CURL, & LAPLACIAN OPERATORS

CARTESIAN (RECTANGULAR) COORDINATES (x, y, z)

$$\nabla V = \mathbf{x}\frac{\partial V}{\partial x} + \mathbf{y}\frac{\partial V}{\partial y} + \mathbf{z}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \mathbf{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

CYLINDRICAL COORDINATES (r, ϕ, z)

$$\nabla V = \mathbf{r}\frac{\partial V}{\partial r} + \boldsymbol{\phi}\frac{1}{r}\frac{\partial V}{\partial \phi} + \mathbf{z}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \mathbf{r} & \boldsymbol{\phi} r & \mathbf{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \mathbf{r}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \boldsymbol{\phi}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \mathbf{z}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

SPHERICAL COORDINATES (R, θ, ϕ)

$$\nabla V = \mathbf{R}\frac{\partial V}{\partial R} + \boldsymbol{\theta}\frac{1}{R}\frac{\partial V}{\partial \theta} + \boldsymbol{\phi}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \mathbf{R} & \boldsymbol{\theta} R & \boldsymbol{\phi} R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix}$$

$$= \mathbf{R}\frac{1}{R\sin\theta}\left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right] + \boldsymbol{\theta}\frac{1}{R}\left[\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R}(RA_\phi)\right] + \boldsymbol{\phi}\frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial \theta}\right]$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$