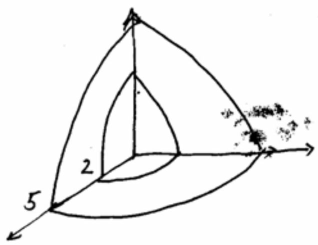
1. (30 points) A section of a sphere is described by $2 \le R \le 5$, $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le \frac{\pi}{2}$. A vector field $\vec{E} = \frac{1}{4\pi R^2}\hat{R}$. Verify divergence theorem by calculating: $\oiint_s \vec{E} \cdot d\vec{s} = \iiint_v (\nabla \cdot \vec{E}) dv$.



$$\nabla \cdot \vec{E} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 E_R) + 0$$

$$= 0$$

$$\iiint_V (\nabla \cdot \vec{E})\, dv = 0$$

$$\oiint \vec{E} \cdot d\vec{s} = \iint_{\text{outer}} \vec{E} \cdot d\vec{s} + \iint_{\text{inner}} \vec{E} \cdot d\vec{s}$$

$$\iint_{\text{outer}} \vec{E} \cdot d\vec{s} = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{1}{4\pi R^2} \cdot R^2 \sin\theta \, d\theta \, d\phi = \frac{1}{4\pi} \cdot \frac{\pi}{2} \cdot 1 = \frac{1}{8}$$

$$\iint_{\text{inner}} \vec{E} \cdot d\vec{s} = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} -\frac{1}{4\pi R^2} R^2 \sin\theta \, d\theta \, d\phi = -\frac{1}{8}$$

$$\oiint \vec{E} \cdot d\vec{s} = \frac{1}{8} - \frac{1}{8} = 0$$

$$\therefore \oiint \vec{E} \cdot d\vec{s} = \iiint (\nabla \cdot \vec{E})\, dV$$

2. (30 points) Determine the line integral of $\vec{F} = 2\hat{x} + z\hat{y} + y\hat{z}$ from P1(-1,3,-2) via P2(2,4,1) to P3 (3,5,6), where points are specified in rectangular coordinate system.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2 & z & y \end{vmatrix} = 0$$

path independent integral

$$\int_{P_1}^{P_3} \vec{F} \cdot d\vec{l} = \int_{P_1(-1,3,-2)}^{A(-1,3,6)} \vec{F} \cdot dl + \int_{A(-1,3,6)}^{B(-1,5,6)} \vec{F} \cdot d\vec{l}$$

$$+ \int_{B(-1,5,6)}^{P_3(3,5,6)} \vec{F} \cdot d\vec{l}$$

$$= \int_{P_1(-1,3,-2)}^{A(-1,3,6)} 2\,\cancel{dx + z\,dy} + y\,dz \Big|_{y=3} + \int_{A(-1,3,6)}^{B(-1,5,6)} z\,dy \Big|_{z=6} + \int_{B(-1,5,6)}^{P_3(3,5,6)} 2\,dx$$

$$= 3 \cdot 8 + 6 \cdot 2 + 4 \cdot 2$$

$$= 44$$

3. (30 points) A If an electric field is given $\vec{E} = 6r\hat{r} + 9r\phi\hat{\phi} - 2r\hat{z}$ (V/m) in the region $0 \le r \le 2,\ -\pi \le \phi \le \pi,\quad 0 \le z \le 2$, assuming $\varepsilon_1 = 2\varepsilon_0 = 1.77\times10^{-11}\, F/m$, determine:

(a) The volume charge density in this region

(b) Total charge in the volume $0 \le r \le 2,\ -\pi \le \phi \le \pi,\quad 0 \le z \le 2$

(a) $\rho_v = \nabla \cdot \vec{D} = \varepsilon \cdot \nabla \cdot \vec{E} = (12\cancel{r} + 9 - 0)\varepsilon$

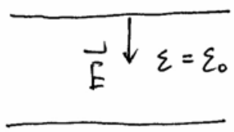
(b) $Q = \iiint \rho_v \cdot dv$

$$= \int_0^{z}\int_0^{2}\int_{-\pi}^{\pi} 21\varepsilon \cdot r\, dr\, dz\, d\phi$$

$$= 21\varepsilon \cdot 2\pi \cdot 2 \cdot \left.\frac{r^2}{2}\right|_0^2 = 8\pi \cdot 21\varepsilon$$

$$= 168\varepsilon.$$

4. (20 points) Determine the maximum total charge can be stored in a parallel plate capacitor. Assume the area of the electrode is 1cm^2. The capacitor is with filled with air. Assume the breakdown ~~voltage~~ $\vec{E}$ of air is 10^7V/m.



$$\vec{E}_{max} = 10^7 \ V/m$$

$$\vec{D}_{max} = \varepsilon \vec{E} = \varepsilon_0 \cdot 10^7$$

$$\rho_{max} = \vec{D} = 10^7 \varepsilon_0$$

$$\Rightarrow Q_{max} = A \cdot \rho_{max} = 1\times10^{-4} \ m^2 \cdot 10^7 \cdot 8.85\times10^{-12}$$

$$= 8.85\times10^{-9} \ (C) = 8.85 \ (nC)$$