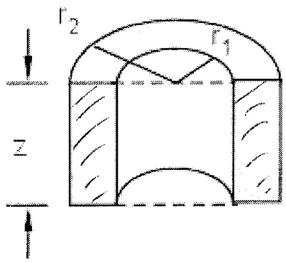
1. (35%) A section of a cylinder is by $1 \le r \le 2$, $0 \le z \le 2$, $0 \le \phi \le \pi$ .shown below, A vector filed is $\vec{E} = \hat{r} r\cos\phi + \hat{\phi} r\sin\phi$. Verify divergence theorem by calculating: $\oint\!\!\oint_s \vec{E}\cdot d\vec{s} = \iiint_v (\nabla\cdot\vec{E})dv$.



Solution.

1. $\nabla\cdot\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}(r\cdot r\cos\phi) + \frac{1}{r}\frac{\partial}{\partial\phi}(r\sin\phi)$

$\nabla\cdot\vec{E} = \frac{1}{r}\cdot 2r\cos\phi + \cos\phi$

$= 3\cos\phi$

$\iiint (\nabla\cdot\vec{E})\cdot dv = \int_1^2\int_0^\pi\int_0^2 3\cos\phi \cdot r\,d\phi\,dr\,dz$

$= 2\cdot\int_1^2 r\,dr \int_0^\pi 3\cos\phi\,d\phi$

$= 6\cdot\frac{r^2}{2}\Big|_1^2 \cdot \sin\phi\Big|_0^\pi = 0$

$\oint\!\!\oint \vec{E}\cdot d\vec{s} = \iint_{top}\vec{E}\cdot d\vec{s} + \iint_{bottom}\vec{E}\cdot ds + \iint_{Side}\vec{E}\cdot d\vec{s} + \iint_{Lef}\vec{E}\cdot d\vec{s} + \iint_{right}\vec{E}\cdot d\vec{s}$

$\iint_{top}\vec{E}\cdot d\vec{s} = \iint(\hat{r}\,r\cos\phi + \hat{\phi}\,r\sin\phi)\cdot\hat{z}\,r\,dr\,d\phi = 0$

$\iint_{bottom}\vec{E}\cdot d\vec{s} = 0$, same as above

$\iint_{Out\ Side}(\hat{r}\,r\cos\phi + \hat{\phi}\sin\phi)\cdot\hat{r}\;r\,d\phi\,dz\Big|_{r=2}$

$$\iint_{\text{Side}} \int_0^2 \int_0^{\pi} r^2 \cos\phi \, d\phi \, dz \Big|_{r=2} = 4 \int_0^2 dz \int_0^{\pi} \cos\phi \, d\phi = 0$$

Same for inside surface.

$$\iint_{\text{Left}} \vec{E} \cdot d\vec{S} = \iint (\hat{r} \cos\phi + \hat{\phi}\, r\sin\phi) \cdot \hat{\phi} \, dr \, dz \Big|_{\phi=\pi}$$

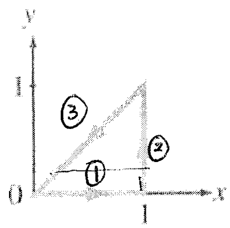
$$= \int_0^2 \int_1^2 r \sin\phi \, dr \, dz \Big|_{\phi=0} = 0$$

$$\iint_{\text{right}} \vec{E} \cdot d\vec{S} = \iint (\hat{r}\, r\cos\phi + \hat{\phi}\, r\sin\phi) \cdot \hat{\phi} \, dr \, dz \Big|_{\phi=0}$$

$$= 0$$

Check out.

2. (35%) Verify Stokes's theorem for the vector field $\vec{A} = \hat{x}y^3 + \hat{y}x^3$ along the contour shown below:



Solution: $\oint \vec{A}\cdot d\vec{l} = \iint (\nabla\times\vec{A})\cdot d\vec{s}$

$$\oint \vec{A}\cdot d\vec{l} = \int_{①} \vec{A}\cdot d\vec{l} + \int_{②} \vec{A}\cdot d\vec{l} + \int_{③} \vec{A}\cdot d\vec{l}$$

$$\int_{①} \vec{A}\cdot d\vec{l} = \int_0^1 (\hat{x}y^3 + \hat{y}x^3)\cdot\hat{x}\,dx\Big|_{y=0} = \int_0^1 y^3\,dx\Big|_{y=0} = 0$$

$$\int_{②} \vec{A}\cdot d\vec{l} = \int_0^1 (\hat{x}y^3 + \hat{y}x^3)\cdot\hat{y}\,dy\Big|_{x=1} = \int_0^1 x^3\,dy\Big|_{x=1} = 1$$

$$\int_{③} \vec{A}\cdot d\vec{l} = \int_0^1 (\hat{x}y^3 + \hat{y}x^3)(-\hat{x}\,dx - \hat{y}\,dy)$$

$$= \int_0^1 -x^3\,dy - y^3\,dx\Big|_{x=y} = -2\int_0^1 x^3\,dx = -2\cdot\frac{x^4}{4}\Big|$$

$$= -\frac{1}{2}$$

$$\boxed{\oint \vec{A}\cdot d\vec{l} = -\frac{1}{2} + 1 = \frac{1}{2}}$$

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 & x^3 & 0 \end{vmatrix} = \hat{z}(3x^2 - 3y^2)$$

$$\iint (\nabla\times\vec{A})\cdot d\vec{s} = \int_0^1 dx \int_0^x dy\cdot(3x^2 - 3y^2)\cdot$$

$$= \int_0^1 \left(3x^3 - y^3\Big|_0^x\right)dx = \int_0^1 2x^3\,dx = 2\cdot\frac{x^4}{4}\Big|_0^1 = \frac{1}{2}$$

3 (30 points) T is a scalar, and $\vec{A}$ is a vector. Prove in (x, y, z) coordinate that

(a) $\nabla \times (\nabla T) = 0$

(b) $\nabla \cdot (\nabla \times \vec{A}) = 0$

(c) $\nabla \cdot (\nabla T) = \dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} + \dfrac{\partial^2 T}{\partial z^2}$

(a). $$\nabla T = \hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial y}$$

$$\nabla \times (\nabla T) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z} \end{vmatrix} = \hat{x}\left[\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial y}\right)\right]$$

$$+ \hat{y}\left[\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial z}\right)\right]$$

$$\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y}\right) \qquad + \hat{z}\left[\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial x}\right)\right]$$

$$= 0$$

(b) $$\nabla \times \vec{A} = \hat{x}\left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right) + \hat{y}\left(\frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z\right) + \hat{z}\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right)$$

$$\nabla \cdot (\nabla \times \vec{A}) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right] + \frac{\partial}{\partial y}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] + \frac{\partial}{\partial z}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

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$$= \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} + \frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial y \partial x} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y}$$

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