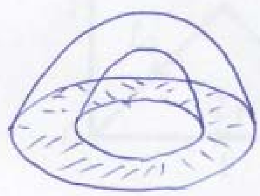
1. (35%) A section of a sphere is described by $2 \le R \le 5$, $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le 2\pi$. A vector field $\vec{E} = \frac{1}{4\pi R^2}\hat{R}$. Verify divergence theorem by calculating: $\oiint_s \vec{E} \cdot d\vec{s} = \iiint_v (\nabla \cdot \vec{E}) dv$.



$$\iint_{\text{inner}} \vec{E} \cdot d\vec{s} = \iint \frac{1}{4\pi R^2} \hat{R} \cdot (-\hat{R}) R\sin\theta \, d\phi \, R \, d\theta \Big|_{R=2}$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} -\frac{1}{4\pi} \sin\theta \, d\theta \, d\phi$$

$$= \frac{+1}{4\pi} \cdot 2\pi \cdot \cos\theta \Big|_0^{\frac{\pi}{2}} = -\frac{1}{2}$$

$$\iint_{\text{outer}} \vec{E} \cdot d\vec{s} = \iint \frac{1}{4\pi R^2} \hat{R} \cdot (\hat{R}) R\sin\theta \, d\phi \, R \, d\theta \Big|_{R=5}$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \frac{1}{4\pi} \sin\theta \, d\theta \, d\phi = \frac{1}{2}$$

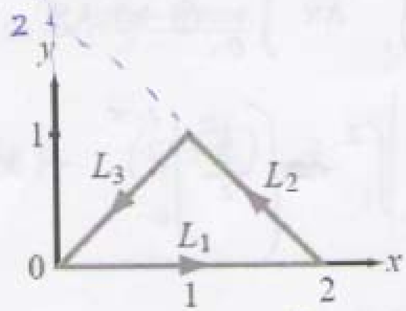
$$\iint_{\text{ring}} \frac{1}{4\pi R^2} \hat{R} \cdot \hat{\theta} \, dR \, R\sin\theta \, d\phi = 0$$

$$\oiint \vec{E} \cdot d\vec{s} = -\frac{1}{2} + \frac{1}{2} + 0 = 0$$

$$\nabla \cdot \vec{E} = 0$$

$$\iiint (\nabla \cdot \vec{E}) \cdot dV = 0$$

2. (35%) Verify Stokes's theorem for the vector field $\vec{A} = \hat{x}xy + \hat{y}yx$ along the path shown below:



$$\oint \vec{A} \cdot \vec{dl} = \iint (\nabla \times \vec{A}) \cdot ds$$

$$\oint \vec{A} \cdot \vec{dl} = \int_1 \vec{A} \cdot \vec{dl} + \int_2 \vec{A} \cdot \vec{dl} + \int_3 \vec{A} \cdot \vec{dl}$$

$$\int_1 \vec{A} \cdot \vec{dl} = \int_0^2 (\hat{x}xy + \hat{y}yx) \cdot \hat{x}\,dx \Big|_{y=0} = 0$$

$$\int_2 \vec{A} \cdot \vec{dl} = \int (\hat{x}xy + \hat{y}yx) \cdot (\hat{x}\,dx + \hat{y}\,dy)$$

$$= -\int_1^2 \hat{x}\,xy\,dx \Big|_{y=2-x} + \int_0^1 yx\,dy \Big|_{x=2-y}$$

$$= -\int_1^2 x(2-x)\,dx + \int_0^1 y(2-y)\,dy$$

$$= -\left(x^2 - \frac{x^3}{3}\right)\Big|_1^2 + \left(y^2 - \frac{y^3}{3}\right)\Big|_0^1 = 0$$

$$\int_3 \vec{A} \cdot \vec{dl} = \int (\hat{x}xy + \hat{y}xy)(-\hat{x}\,dx - \hat{y}\,dy)$$

$$= -\int_0^1 xy\,dx - \int_0^1 xy\,dy \Big|_{x=y}$$

$$= -2\int_0^1 x^2\,dx = -2 \cdot \frac{x^3}{3}\Big|_0^1 = -\frac{2}{3}$$

$$(\nabla \times \vec{A}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \hat{x}\,0 + \hat{y}\,0 + \hat{z}(y - x)$$

$$\iint (\nabla \times \vec{A}) \cdot \vec{ds} = \iint (y-x)\, \hat{z} \cdot \hat{z}\, dx\, dy$$

$$= \int_0^1 dx \int_0^x (y-x)\, dy + \int_1^2 dx \int_0^{2-x} (y-x)\, dy$$

$$= \int_0^1 dx \cdot \left[\frac{y^2}{2} \Big|_0^x - x^2 \right] + \int_1^2 \left[\frac{y^2}{2} \Big|_0^{2-x} - x(2-x) \right] dx$$

$$= \int_0^1 dx \left(-\frac{x^2}{2}\right) + \int_1^2 \left(2 - 4x + \frac{3x^2}{2}\right) dx$$

$$= -\frac{x^3}{6} \Big|_0^1 + 2\cdot(1) - 4 \cdot \frac{x^2}{2} \Big|_1^2 + \frac{x^3}{2} \Big|_1^2$$

$$= -\frac{1}{6} + 2 - 6 + \frac{7}{2}$$

$$= -\frac{1}{6} - \frac{1}{2}$$

$$= -\frac{2}{3}$$

$$\therefore \quad \iint (\nabla \times \vec{A}) \cdot \vec{ds} = \oint \vec{A} \cdot \vec{dl}$$

3 (30 points) Determine whether each of the following vector field is conservative:

(a) $\vec{A} = \hat{x}x^2 - \hat{y}y^2$

(b) $\vec{B} = \hat{r}\frac{\sin\phi}{r^2} - \hat{\phi}\frac{\cos\phi}{r^2}$

(a) $$(\nabla \times \vec{A}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & 0 \end{vmatrix}$$

$$= \hat{x}\,0 + \hat{y}\,0 + \hat{z}\,0$$

conservative

(b) $$(\nabla \times \vec{B}) = \begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \frac{\sin\phi}{r^2} & -\frac{\cos\phi}{r^2} & 0 \end{vmatrix}$$

$$= \hat{r}\,0 + \hat{\phi}\,0 + \hat{z}\left[\frac{\partial}{\partial r}\left(-\frac{r\cos\phi}{r^2}\right) - \frac{\partial}{\partial \phi}\left(\frac{\sin\phi}{r^2}\right)\right]$$

$$= \hat{r}\,0 + \hat{\phi}\,0 + \hat{z}\left[(-1)\frac{\cos\phi}{r^2} + \frac{\cos\phi}{r^2}\right]$$

$$= 0$$

conservative