1. (10 points) On a lossless 50-Ω transmission line terminated with a $Z_L = 100+j40\ \Omega$. If this transmission line is to be matched to the load using a shorted load stub. Determine the stub length and distance between the load and stub. Two possible answers. You only need to show one of them.

2. (10 points) A 50cm long air spaced lossless 50-Ω line ($\varepsilon_r = 1$) is terminated in an unknown impedance. If the input impedance is Zin = 40 + j30-Ω at frequency 5GHz, Find (1) the reflection coefficient; (2) the standing wave ratio (S) and (3) the location of the first voltage maximum from the load.

   (1) $\Gamma = 0.25\angle{-35^0}$

   (2) S = 2.0

   (3) 0.455 λ

(1) The wavelength is $\lambda = \frac{c}{f} = 6cm$. The length of the transmission line is 8.33λ, which is equivalent to 0.33λ.

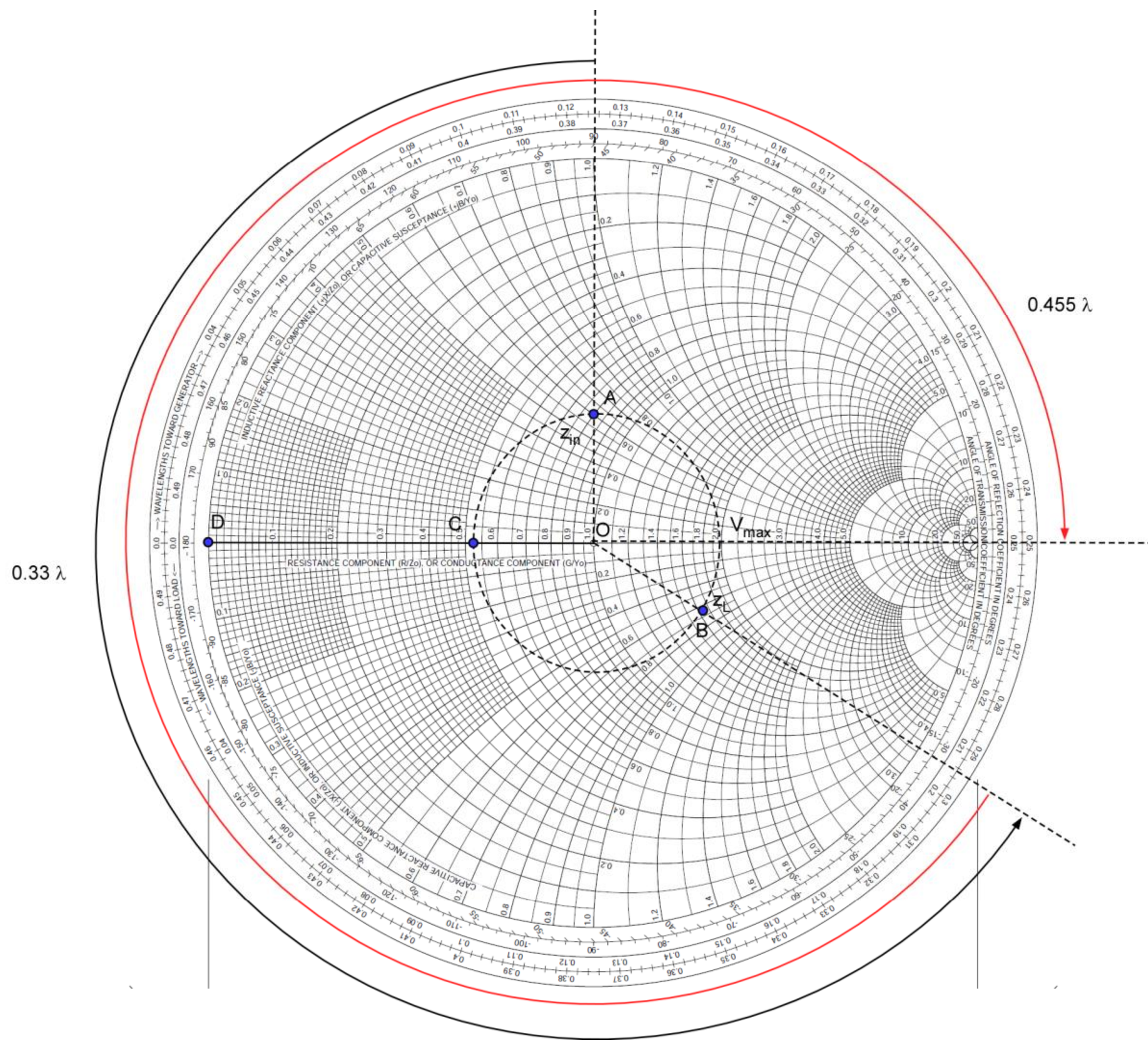
$z_{in} = \frac{Z_{in}}{Z_0} = 0.8 + j0.6$, which is point A on Smith chart.

Z_L is point B on Smith chart. $\angle{-35^0}$

$$|\Gamma| = \frac{|OC|}{|OD|} = 0.25$$

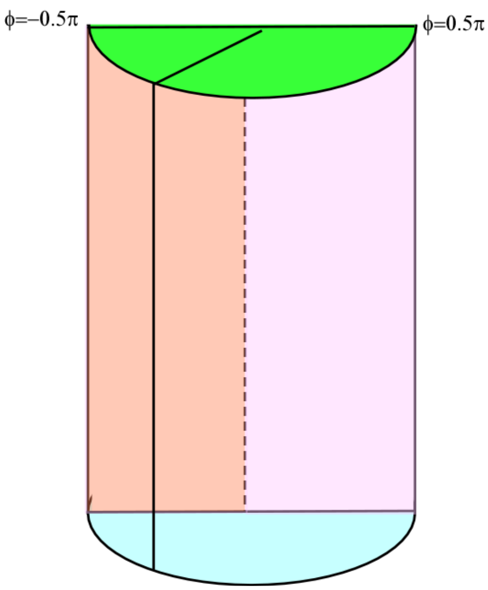
(2) S = 2.0

(3) 0.455 λ


0.455 λ
0.33 λ
A
z_{in}
D
C
O
V_{max}
z_L
B
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
ANGLE OF TRANSMISSION COEFFICIENT IN DEGREES

3. (10 points) For a vector filed $\vec{A} = 3r^2\hat{r} + 3r\phi\hat{\phi} - 2\hat{z}$, verify the divergence theorem $\oiint_s \vec{A}\cdot d\vec{s} = \iiint_v (\nabla\cdot\vec{A})dv$, on a section of a cylinder bounded by

$r = 1, \ -\frac{\pi}{2} \le \phi \le \frac{\pi}{2}, \ \ 1 \le z \le 3.$



Solution:

(1) $$\oiint_s \vec{A}\cdot d\vec{s} = \iint_{top} \vec{A}\cdot d\vec{s} + \iint_{bottom} \vec{A}\cdot d\vec{s} + \iint_{outter} \vec{A}\cdot d\vec{s} + \iint_{left} \vec{A}\cdot d\vec{s} \iint_{right} \vec{A}\cdot d\vec{s}$$

$$\iint_{top} \vec{A}\cdot d\vec{s} = \iint_{top} -2rdrd\phi\,|_{z=3} = -\iint_{bottom} \vec{A}\cdot d\vec{s}$$

$$\iint_{outter} \vec{A}\cdot d\vec{s} = \iint_{outter} 3r^2 rd\phi dz\,|_{r=1} = \int_1^3 dz \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3d\phi = 6\pi$$

$$\iint_{left} \vec{A}\cdot d\vec{s} = -\iint_{left} 3r\phi drdz\,|_{\phi=-\frac{\pi}{2}} = \frac{3\pi}{2}\int_1^3 dz \int_0^1 rdr = \frac{3\pi}{2}$$

$$\iint_{right} \vec{A}\cdot d\vec{s} = \iint_{right} 3r\phi drdz\,|_{\phi=\frac{\pi}{2}} = \frac{3\pi}{2}\int_1^3 dz \int_0^1 rdr = \frac{3\pi}{2}$$

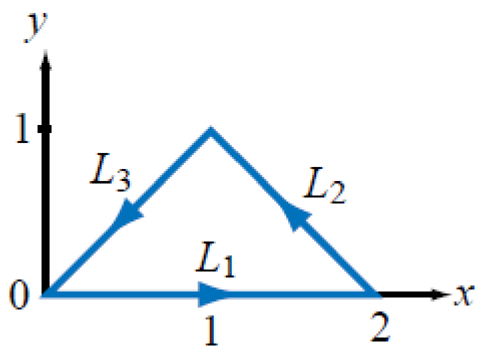
$$\oiint_s \vec{A}\cdot d\vec{s} = 9\pi$$

(2) $$\nabla\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(r3r^2) + \frac{1}{r}\frac{\partial}{\partial \phi}(3r\phi) + \frac{\partial}{\partial z}(-2) = 9r + 3$$

$$\iiint_v (\nabla \cdot \vec{A}) dv = \int_1^3 dz \int_0^1 dr \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r(9r+3) d\phi$$

$$= 2\pi \int_0^1 r(9r+3) dr = 9\pi$$

4. (10 points) Verify Stokes's theorem for the vector field $\vec{A} = \hat{x}2xy + \hat{y}yx$ along the path shown below:



See Hw solution

5. (10 points) If an electric field is given $\vec{E} = 6r\hat{r} + 9r\phi\hat{\phi} - 2r\hat{z}$ in the region $0 \le r \le 2,\ -\pi \le \phi \le \pi,\quad 0 \le z \le 4$, determine:

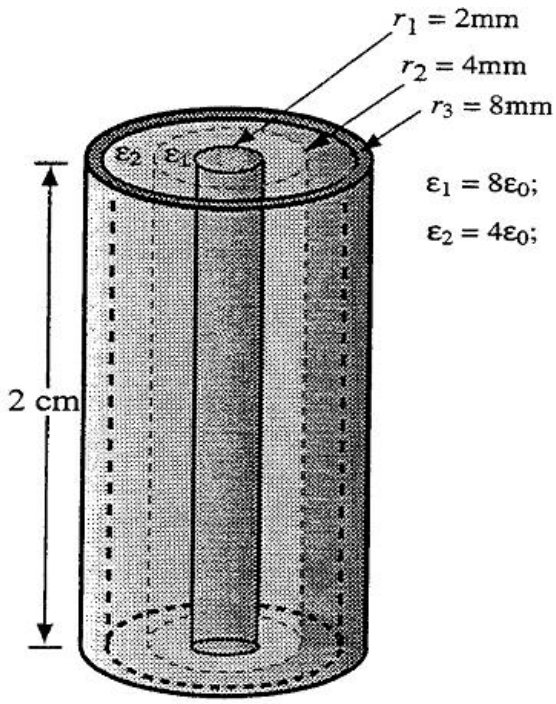
(1) The volume charge density in this region

(2) Total charge in the volume $0 \le r \le 2,\ -\pi \le \phi \le \pi,\quad 0 \le z \le 4$.

(1) $\rho = \nabla \cdot \vec{D} = \varepsilon \nabla \cdot \vec{E} = 21\varepsilon$

(2) $Q = \rho V = 21\varepsilon\pi 16$

6. (10 points) A coaxial capacitor with inner connector radius r_1 = 2mm, and outer connector radius r_3= 8mm, is filled with two different materials as shown in the following figure. The length of the capacitor is 2cm. Calculate the capacitance of the capacitor.



Solution:

(1) Assuming surface charge density in the inner conductor is σ1, then

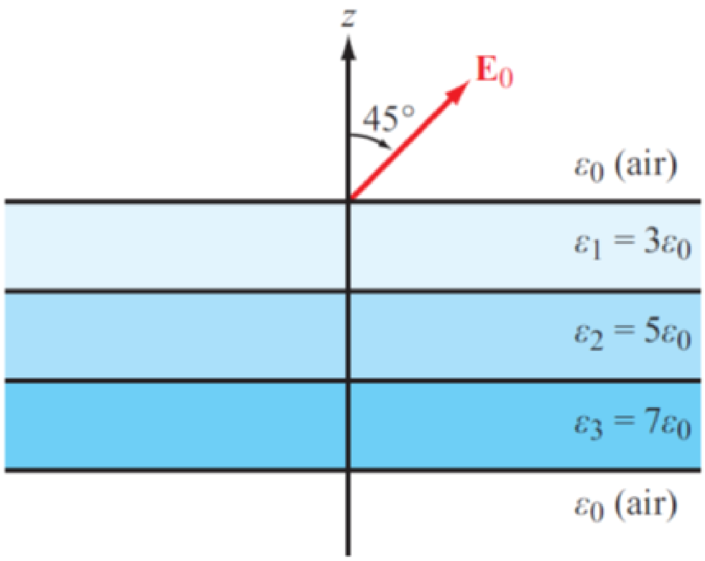
$$\vec{D} = \hat{r}\frac{\sigma_1 r_1}{r}$$, (r1<r<r3),

(2) $\vec{E}_1 = \hat{r}\frac{\sigma_1 r_1}{\varepsilon_1 r}$ (r1<r<r2), $\vec{E}_2 = \hat{r}\frac{\sigma_1 r_1}{\varepsilon_2 r}$ (r2<r<r3),

(3) $$V = \frac{\sigma_1 r_1}{\varepsilon_1}\ln\frac{r_2}{r_1} + \frac{\sigma_1 r_1}{\varepsilon_2}\ln\frac{r_3}{r_2}$$

(4) $$C = \frac{L}{2\pi}\left(\frac{1}{\varepsilon_1}\ln\frac{r_2}{r_1} + \frac{1}{\varepsilon_2}\ln\frac{r_3}{r_2}\right)$$

7. (10 points) Determine the electric field intensity vectors in each layer.



Solutions:

Layer ε_1:

$$E_{1t} = E_{0t} = \frac{\sqrt{2}}{2}E_0,\ \varepsilon_1 E_{1z} = \varepsilon_0 E_{0z},\ E_{1z} = \frac{\varepsilon_0}{\varepsilon_1}E_{0z} = \frac{\sqrt{2}}{6}E_0,$$

Layer ε_2:

$$E_{2t} = E_{1t} = \frac{\sqrt{2}}{2}E_0,\ \varepsilon_2 E_{2z} = \varepsilon_1 E_{1z} = \varepsilon_0 E_{0z},\ E_{2z} = \frac{\varepsilon_0}{\varepsilon_2}E_{0z} = \frac{\sqrt{2}}{10}E_0,$$

Layer ε_3:

$$E_{3t} = E_{2t} = \frac{\sqrt{2}}{2}E_0,\ \varepsilon_3 E_{3z} = \varepsilon_2 E_{2z} = \varepsilon_0 E_{0z},\ E_{3z} = \frac{\varepsilon_0}{\varepsilon_3}E_{0z} = \frac{\sqrt{2}}{14}E_0,$$

Layer ε_0:

$$E_{4t} = E_{3t} = \frac{\sqrt{2}}{2}E_0,\ \varepsilon_0 E_{4z} = \varepsilon_3 E_{3z} = \varepsilon_0 E_{0z},\ E_{4z} = \frac{\varepsilon_0}{\varepsilon_0}E_{0z} = \frac{\sqrt{2}}{2}E_0.$$

8. (10 points) In a certain conducting region, the magnetic field is given in cylindrical coordinates by $\vec{H} = \hat{\phi}\frac{4}{r}\left[1-(1+2r)e^{-2r}\right]$. Find out the current density $\vec{J}$.

$$J = \nabla \times \vec{H} = \hat{z}\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{4}{r}\left[1-(1+2r)e^{-2r}\right]\right\} = \hat{z}\frac{1}{r}\frac{\partial}{\partial r}\left\{4\left[1-(1+2r)e^{-2r}\right]\right\} = 4\frac{1}{r}\hat{z}\left\{2(1+2r)e^{-2r} - 2e^{-2r}\right\}$$

$$= \hat{z}16e^{-2r}$$

9. (10 points) Write down the Maxwell equations and their integral forms.

See class notes

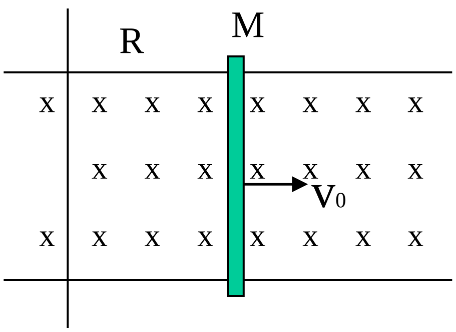
10. (10 points) A conducting bar is put in a constant magnetic field B = 0.1T. The circuit resistance is R =20Ω. The bar width is 20cm. The mass of the bar is m = 1kg. If the bar has an initial speed (t = 0) of V = 4m/s, determine

(1) the current generated in the bar at t = 0;

(2) the force experienced by the conducting bar at t = 0;

(3) (extra 5 points) the speed of the bar at time t = 10s;

(4) (extra 5 points) the current generated at t =10s?



(1) I = 4(mA), counter clockwise

(2) F = 8×10^{-5} (N), pointing to the left

(3) $F = -m\frac{d}{dt}v = \frac{B^2W^2}{R}v$,

$$v = v_0 e^{-\frac{W^2B^2}{mR}t}$$

(4) $I = \frac{WBv_0}{R} e^{-\frac{W^2B^2}{mR}t}$

GRADIENT, DIVERGENCE, CURL, & LAPLACIAN OPERATORS

CARTESIAN (RECTANGULAR) COORDINATES (x, y, z)

$$\nabla V = \mathbf{x}\frac{\partial V}{\partial x} + \mathbf{y}\frac{\partial V}{\partial y} + \mathbf{z}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \mathbf{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

CYLINDRICAL COORDINATES (r, ϕ, z)

$$\nabla V = \mathbf{r}\frac{\partial V}{\partial r} + \boldsymbol{\phi}\frac{1}{r}\frac{\partial V}{\partial \phi} + \mathbf{z}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \mathbf{r} & \boldsymbol{\phi} r & \mathbf{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \mathbf{r}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \boldsymbol{\phi}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \mathbf{z}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

SPHERICAL COORDINATES (R, θ, ϕ)

$$\nabla V = \mathbf{R}\frac{\partial V}{\partial R} + \boldsymbol{\theta}\frac{1}{R}\frac{\partial V}{\partial \theta} + \boldsymbol{\phi}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \mathbf{R} & \boldsymbol{\theta} R & \boldsymbol{\phi} R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix}$$

$$= \mathbf{R}\frac{1}{R\sin\theta}\left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right] + \boldsymbol{\theta}\frac{1}{R}\left[\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R}(RA_\phi)\right] + \boldsymbol{\phi}\frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial \theta}\right]$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

Table 2-3: Properties of standing waves on a lossless transmission line.

$S = \frac{|\Gamma|+1}{1-|\Gamma|}$

Voltage maximum Voltage minimum	$\lvert\widetilde{V}\rvert_{\max} = \lvert V_0^+\rvert[1+\lvert\Gamma\rvert]$ $\lvert\widetilde{V}\rvert_{\min} = \lvert V_0^+\rvert[1-\lvert\Gamma\rvert]$ $\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$
Positions of voltage maxima (also positions of current minima) Position of first maximum (also position of first current minimum)	$l_{\max} = \frac{\theta_r \lambda}{4\pi} + \frac{n\lambda}{2}, \quad n = 0, 1, 2, \ldots$ $l_{\max} = \begin{cases} \frac{\theta_r \lambda}{4\pi}, & \text{if } 0 \le \theta_r \le \pi \\ \frac{\theta_r \lambda}{4\pi} + \frac{\lambda}{2}, & \text{if } -\pi \le \theta_r \le 0 \end{cases}$
Positions of voltage minima (also positions of first current maxima) Position of first minimum (also position of first current maximum)	$l_{\min} = \frac{\theta_r \lambda}{4\pi} + \frac{(2n+1)\lambda}{4}, \quad n = 0, 1, 2, \ldots$ $l_{\min} = \frac{\lambda}{4}\left(1 + \frac{\theta_r}{\pi}\right)$
Input impedance	$Z_{in} = Z_0 \left(\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \right)$
Positions at which Z_{in} is real	at voltage maxima and minima
Z_{in} at voltage maxima	$Z_{in} = Z_0 \left(\frac{1+\lvert\Gamma\rvert}{1-\lvert\Gamma\rvert} \right)$
Z_{in} at voltage minima	$Z_{in} = Z_0 \left(\frac{1-\lvert\Gamma\rvert}{1+\lvert\Gamma\rvert} \right)$
Z_{in} of short-circuited line	$Z_{in}^{sc} = jZ_0 \tan \beta l$
Z_{in} of open-circuited line	$Z_{in}^{oc} = -jZ_0 \cot \beta l$
Z_{in} of line of length $l = n\lambda/2$	$Z_{in} = Z_L, \quad n = 0, 1, 2, \ldots$
Z_{in} of line of length $l = \lambda/4 + n\lambda/2$ Z_{in} of matched line	$Z_{in} = Z_0^2/Z_L, \quad n = 0, 1, 2, \ldots$ $Z_{in} = Z_0$

$\lvert V_0^+\rvert$ = amplitude of incident wave, $\Gamma = \lvert\Gamma\rvert e^{j\theta_r}$ with $-\pi < \theta_r < \pi$; θ_r in radians.

Table 3-1: Summary of vector relations.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation, A =	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A, $\lvert A\rvert$ =	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1}$ =	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product, A · B =	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product, A × B =	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length, $d\mathbf{l}$ =	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume, $d\mathcal{V}$ =	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Boundary conditions:

$$D_{1n} - D_{2n} = \rho_s$$

$$H_{1x} - H_{2x} = J_y$$

$$H_{1y} - H_{2y} = -J_x$$

EE 16.360 Final Exam

12/17/2010

Name: ______________________________

Signature: ______________________________