Problem 3.39 For the vector field $\mathbf{E} = \hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy$, verify the divergence theorem by computing:

(a) the total outward flux flowing through the surface of a cube centered at the origin and with sides equal to 2 units each and parallel to the Cartesian axes, and

(b) the integral of $\nabla \cdot \mathbf{E}$ over the cube's volume.

Solution:

(a) For a cube, the closed surface integral has 6 sides:

$$\oint \mathbf{E} \cdot d\mathbf{s} = F_{\text{top}} + F_{\text{bottom}} + F_{\text{right}} + F_{\text{left}} + F_{\text{front}} + F_{\text{back}},$$

$$F_{\text{top}} = \int_{x=-1}^{1} \int_{y=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\Big|_{z=1} \cdot (\hat{\mathbf{z}}\,dy\,dx)$$

$$= -\int_{x=-1}^{1} \int_{y=-1}^{1} xy\,dy\,dx = \left(\left(\frac{x^2y^2}{4}\right)\Big|_{y=-1}^{1}\right)\Bigg|_{x=-1}^{1} = 0,$$

$$F_{\text{bottom}} = \int_{x=-1}^{1} \int_{y=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\Big|_{z=-1} \cdot (-\hat{\mathbf{z}}\,dy\,dx)$$

$$= \int_{x=-1}^{1} \int_{y=-1}^{1} xy\,dy\,dx = \left(\left(\frac{x^2y^2}{4}\right)\Big|_{y=-1}^{1}\right)\Bigg|_{x=-1}^{1} = 0,$$

$$F_{\text{right}} = \int_{x=-1}^{1} \int_{z=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\Big|_{y=1} \cdot (\hat{\mathbf{y}}\,dz\,dx)$$

$$= -\int_{x=-1}^{1} \int_{z=-1}^{1} z^2\,dz\,dx = -\left(\left(\frac{xz^3}{3}\right)\Big|_{z=-1}^{1}\right)\Bigg|_{x=-1}^{1} = \frac{-4}{3},$$

$$F_{\text{left}} = \int_{x=-1}^{1} \int_{z=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\Big|_{y=-1} \cdot (-\hat{\mathbf{y}}\,dz\,dx)$$

$$= -\int_{x=-1}^{1} \int_{z=-1}^{1} z^2\,dz\,dx = -\left(\left(\frac{xz^3}{3}\right)\Big|_{z=-1}^{1}\right)\Bigg|_{x=-1}^{1} = \frac{-4}{3},$$

$$F_{\text{front}} = \int_{y=-1}^{1} \int_{z=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\Big|_{x=1} \cdot (\hat{\mathbf{x}}\,dz\,dy)$$

$$= \int_{y=-1}^{1} \int_{z=-1}^{1} z\,dz\,dy = \left(\left(\frac{yz^2}{2}\right)\Big|_{z=-1}^{1}\right)\Bigg|_{y=-1}^{1} = 0,$$

$$F_{\text{back}} = \int_{y=-1}^{1} \int_{z=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\Big|_{x=-1} \cdot (-\hat{\mathbf{x}}\,dz\,dy)$$

$$= \int_{y=-1}^{1} \int_{z=-1}^{1} z\,dz\,dy = \left(\left(\frac{yz^2}{2} \right) \bigg|_{z=-1}^{1} \right) \bigg|_{y=-1}^{1} = 0,$$

$$\oint \mathbf{E} \cdot d\mathbf{s} = 0 + 0 + \frac{-4}{3} + \frac{-4}{3} + 0 + 0 = \frac{-8}{3}\,.$$

(b)

$$\iiint \nabla \cdot \mathbf{E}\,dv = \int_{x=-1}^{1} \int_{y=-1}^{1} \int_{z=-1}^{1} \nabla \cdot (\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy)\,dz\,dy\,dx$$

$$= \int_{x=-1}^{1} \int_{y=-1}^{1} \int_{z=-1}^{1} (z - z^2)\,dz\,dy\,dx$$

$$= \left(\left(\left(xy \left(\frac{z^2}{2} - \frac{z^3}{3} \right) \right) \bigg|_{z=-1}^{1} \right) \bigg|_{y=-1}^{1} \right) \bigg|_{x=-1}^{1} = \frac{-8}{3}\,.$$

Problem 3.40 For the vector field $\mathbf{E} = \hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z$, verify the divergence theorem for the cylindrical region enclosed by $r = 2$, $z = 0$, and $z = 4$.

Solution:

$$\begin{aligned}
\oint \mathbf{E}\cdot d\mathbf{s} &= \int_{r=0}^{2}\int_{\phi=0}^{2\pi} \left((\hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z)\cdot(-\hat{\mathbf{z}}r\,dr\,d\phi)\right)\Big|_{z=0} \\
&\quad + \int_{\phi=0}^{2\pi}\int_{z=0}^{4} \left((\hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z)\cdot(\hat{\mathbf{r}}r\,d\phi\,dz)\right)\Big|_{r=2} \\
&\quad + \int_{r=0}^{2}\int_{\phi=0}^{2\pi} \left((\hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z)\cdot(\hat{\mathbf{z}}r\,dr\,d\phi)\right)\Big|_{z=4} \\
&= 0 + \int_{\phi=0}^{2\pi}\int_{z=0}^{4} 10e^{-2}2\,d\phi\,dz + \int_{r=0}^{2}\int_{\phi=0}^{2\pi} -12r\,dr\,d\phi \\
&= 160\pi e^{-2} - 48\pi \approx -82.77, \\
\iiint \nabla\cdot\mathbf{E}\,d\mathcal{V} &= \int_{z=0}^{4}\int_{r=0}^{2}\int_{\phi=0}^{2\pi} \left(\frac{10e^{-r}(1-r)}{r} - 3\right) r\,d\phi\,dr\,dz \\
&= 8\pi\int_{r=0}^{2} \left(10e^{-r}(1-r) - 3r\right)dr \\
&= 8\pi\left(-10e^{-r} + 10e^{-r}(1+r) - \frac{3r^2}{2}\right)\Bigg|_{r=0}^{2} \\
&= 160\pi e^{-2} - 48\pi \approx -82.77.
\end{aligned}$$

Problem 3.41 A vector field $\mathbf{D} = \hat{\mathbf{r}}r^3$ exists in the region between two concentric cylindrical surfaces defined by $r = 1$ and $r = 2$, with both cylinders extending between $z = 0$ and $z = 5$. Verify the divergence theorem by evaluating:

(a) $\oint_S \mathbf{D} \cdot d\mathbf{s}$,

(b) $\int_{\mathscr{V}} \nabla \cdot \mathbf{D}\, d\mathscr{V}$.

Solution:

(a)

$$\iint \mathbf{D} \cdot d\mathbf{s} = F_{\text{inner}} + F_{\text{outer}} + F_{\text{bottom}} + F_{\text{top}},$$

$$F_{\text{inner}} = \int_{\phi=0}^{2\pi} \int_{z=0}^{5} \left((\hat{\mathbf{r}}r^3) \cdot (-\hat{\mathbf{r}}r\, dz\, d\phi) \right)\Big|_{r=1}$$

$$= \int_{\phi=0}^{2\pi} \int_{z=0}^{5} \left(-r^4\, dz\, d\phi \right)\Big|_{r=1} = -10\pi,$$

$$F_{\text{outer}} = \int_{\phi=0}^{2\pi} \int_{z=0}^{5} \left((\hat{\mathbf{r}}r^3) \cdot (\hat{\mathbf{r}}r\, dz\, d\phi) \right)\Big|_{r=2}$$

$$= \int_{\phi=0}^{2\pi} \int_{z=0}^{5} \left(r^4\, dz\, d\phi \right)\Big|_{r=2} = 160\pi,$$

$$F_{\text{bottom}} = \int_{r=1}^{2} \int_{\phi=0}^{2\pi} \left((\hat{\mathbf{r}}r^3) \cdot (-\hat{\mathbf{z}}r\, d\phi\, dr) \right)\Big|_{z=0} = 0,$$

$$F_{\text{top}} = \int_{r=1}^{2} \int_{\phi=0}^{2\pi} \left((\hat{\mathbf{r}}r^3) \cdot (\hat{\mathbf{z}}r\, d\phi\, dr) \right)\Big|_{z=5} = 0.$$

Therefore, $\iint \mathbf{D} \cdot d\mathbf{s} = 150\pi$.

(b) From the back cover, $\nabla \cdot \mathbf{D} = (1/r)(\partial/\partial r)(rr^3) = 4r^2$. Therefore,

$$\iiint \nabla \cdot \mathbf{D}\, d\mathscr{V} = \int_{z=0}^{5} \int_{\phi=0}^{2\pi} \int_{r=1}^{2} 4r^2 r\, dr\, d\phi\, dz = \left(\left((r^4)\Big|_{r=1}^{2} \right)\Big|_{\phi=0}^{2\pi} \right)\Big|_{z=0}^{5} = 150\pi.$$