Problem 3.43 For the vector fiel $\mathbf{E} = \hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2 + 2y^2)$, calculate

(a) $\oint_C \mathbf{E} \cdot d\mathbf{l}$ around the triangular contour shown in Fig. P3.50(a), and

(b) $\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{s}$ over the area of the triangle.

Solution: In addition to the independent condition that $z = 0$, the three lines of the triangle are represented by the equations $y = 0$, $x = 1$, and $y = x$, respectively.

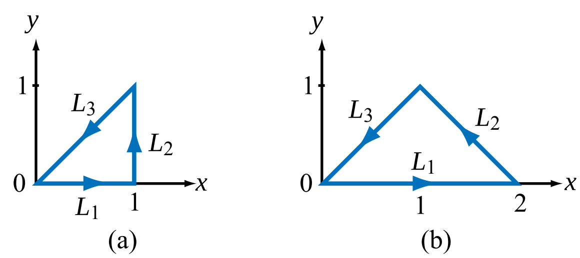

Figure P3.43: Contours for (a) Problem 3.43 and (b) Problem 3.44.

(a)

$$\oint \mathbf{E} \cdot d\mathbf{l} = L_1 + L_2 + L_3,$$

$$L_1 = \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2 + 2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz)$$

$$= \int_{x=0}^{1} (xy)|_{y=0,z=0}\,dx - \int_{y=0}^{0} (x^2 + 2y^2)\big|_{z=0}\,dy + \int_{z=0}^{0} (0)|_{y=0}\,dz = 0,$$

$$L_2 = \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2 + 2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz)$$

$$= \int_{x=1}^{1} (xy)|_{z=0}\,dx - \int_{y=0}^{1} (x^2 + 2y^2)\big|_{x=1,z=0}\,dy + \int_{z=0}^{0} (0)|_{x=1}\,dz$$

$$= 0 - \left(y + \frac{2y^3}{3}\right)\bigg|_{y=0}^{1} + 0 = \frac{-5}{3},$$

$$L_3 = \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2 + 2y^2)) \cdot (\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz)$$

$$= \int_{x=1}^{0} (xy)|_{y=x,\ z=0}\,dx - \int_{y=1}^{0} (x^2 + 2y^2)\big|_{x=y,\ z=0}\,dy + \int_{z=0}^{0} (0)|_{y=x}\,dz$$

$$= \left(\frac{x^3}{3}\right)\bigg|_{x=1}^{0} - \left(y^3\right)\big|_{y=1}^{0} + 0 = \frac{2}{3}.$$

Therefore,

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 - \frac{5}{3} + \frac{2}{3} = -1.$$

(b) From Eq. (3.105), $\nabla \times \mathbf{E} = -\hat{\mathbf{z}}3x$, so that

$$\iint \nabla \times \mathbf{E} \cdot d\mathbf{s} = \int_{x=0}^{1} \int_{y=0}^{x} ((-\hat{\mathbf{z}}3x) \cdot (\hat{\mathbf{z}}\, dy\, dx))\big|_{z=0}$$

$$= -\int_{x=0}^{1} \int_{y=0}^{x} 3x\, dy\, dx = -\int_{x=0}^{1} 3x(x-0)\, dx = -\left(x^3\right)\big|_0^1 = -1.$$

Problem 3.44 Repeat Problem 3.43 for the contour shown in Fig. P3.43(b).

Solution: In addition to the independent condition that $z = 0$, the three lines of the triangle are represented by the equations $y = 0$, $y = 2 - x$, and $y = x$, respectively.

(a)

$$\oint \mathbf{E} \cdot d\mathbf{l} = L_1 + L_2 + L_3,$$

$$\begin{aligned}
L_1 &= \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2 + 2y^2)) \cdot (\hat{\mathbf{x}}\, dx + \hat{\mathbf{y}}\, dy + \hat{\mathbf{z}}\, dz) \\
&= \int_{x=0}^{2} (xy)|_{y=0,z=0}\, dx - \int_{y=0}^{0} \left(x^2 + 2y^2\right)\big|_{z=0}\, dy + \int_{z=0}^{0} (0)|_{y=0}\, dz = 0, \\
L_2 &= \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2 + 2y^2)) \cdot (\hat{\mathbf{x}}\, dx + \hat{\mathbf{y}}\, dy + \hat{\mathbf{z}}\, dz) \\
&= \int_{x=2}^{1} (xy)|_{z=0,y=2-x}\, dx - \int_{y=0}^{1} \left(x^2 + 2y^2\right)\big|_{x=2-y,z=0}\, dy + \int_{z=0}^{0} (0)|_{y=2-x}\, dz \\
&= \left(x^2 - \frac{x^3}{3}\right)\bigg|_{x=2}^{1} - \left(4y - 2y^2 + y^3\right)\big|_{y=0}^{1} + 0 = \frac{-11}{3}, \\
L_3 &= \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2 + 2y^2)) \cdot (\hat{\mathbf{x}}\, dx + \hat{\mathbf{y}}\, dy + \hat{\mathbf{z}}\, dz) \\
&= \int_{x=1}^{0} (xy)|_{y=x,z=0}\, dx - \int_{y=1}^{0} \left(x^2 + 2y^2\right)\big|_{x=y,z=0}\, dy + \int_{z=0}^{0} (0)|_{y=x}\, dz \\
&= \left(\frac{x^3}{3}\right)\bigg|_{x=1}^{0} - \left(y^3\right)\big|_{y=1}^{0} + 0 = \frac{2}{3}.
\end{aligned}$$

Therefore,

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 - \frac{11}{3} + \frac{2}{3} = -3.$$

(b) From Eq. (3.105), $\nabla \times \mathbf{E} = -\hat{\mathbf{z}}3x$, so that

$$\begin{aligned}
\iint \nabla \times \mathbf{E} \cdot d\mathbf{s} &= \int_{x=0}^{1} \int_{y=0}^{x} \left((-\hat{\mathbf{z}}3x) \cdot (\hat{\mathbf{z}}\, dy\, dx)\right)|_{z=0} \\
&\quad + \int_{x=1}^{2} \int_{y=0}^{2-x} \left((-\hat{\mathbf{z}}3x) \cdot (\hat{\mathbf{z}}\, dy\, dx)\right)|_{z=0} \\
&= -\int_{x=0}^{1} \int_{y=0}^{x} 3x\, dy\, dx - \int_{x=1}^{2} \int_{y=0}^{2-x} 3x\, dy\, dx \\
&= -\int_{x=0}^{1} 3x(x - 0)\, dx - \int_{x=1}^{2} 3x((2 - x) - 0)\, dx \\
&= -\left(x^3\right)\big|_{0}^{1} - \left(3x^2 - x^3\right)\big|_{x=1}^{2} = -3.
\end{aligned}$$

Problem 3.45 Verify Stokes's theorem for the vector fiel $\mathbf{B} = (\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)$ by evaluating:

(a) $\oint_C \mathbf{B}\cdot d\mathbf{l}$ over the semicircular contour shown in Fig. P3.45(a), and

(b) $\int_S (\nabla\times\mathbf{B})\cdot d\mathbf{s}$ over the surface of the semicircle.

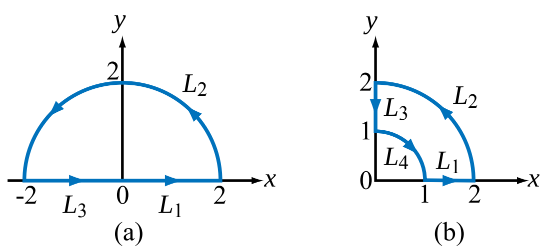

Figure P3.45: Contour paths for (a) Problem 3.45 and (b) Problem 3.46.

Solution:

(a)

$$\oint \mathbf{B}\cdot d\mathbf{l} = \int_{L_1} \mathbf{B}\cdot d\mathbf{l} + \int_{L_2} \mathbf{B}\cdot d\mathbf{l} + \int_{L_3} \mathbf{B}\cdot d\mathbf{l},$$

$$\mathbf{B}\cdot d\mathbf{l} = (\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)\cdot(\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}} r\,d\phi + \hat{\mathbf{z}}\,dz) = r\cos\phi\,dr + r\sin\phi\,d\phi,$$

$$\int_{L_1} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=0}^{2} r\cos\phi\,dr\right)\bigg|_{\phi=0,\ z=0} + \left(\int_{\phi=0}^{0} r\sin\phi\,d\phi\right)\bigg|_{z=0}$$

$$= \left(\tfrac{1}{2}r^2\right)\Big|_{r=0}^{2} + 0 = 2,$$

$$\int_{L_2} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{2} r\cos\phi\,dr\right)\bigg|_{z=0} + \left(\int_{\phi=0}^{\pi} r\sin\phi\,d\phi\right)\bigg|_{r=2,\ z=0}$$

$$= 0 + (-2\cos\phi)\big|_{\phi=0}^{\pi} = 4,$$

$$\int_{L_3} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{0} r\cos\phi\,dr\right)\bigg|_{\phi=\pi, z=0} + \left(\int_{\phi=\pi}^{\pi} r\sin\phi\,d\phi\right)\bigg|_{z=0}$$

$$= \left(-\tfrac{1}{2}r^2\right)\Big|_{r=2}^{0} + 0 = 2,$$

$$\oint \mathbf{B}\cdot d\mathbf{l} = 2+4+2 = 8.$$

(b)

$$\begin{aligned}
\nabla\times\mathbf{B} &= \nabla\times(\hat{\mathbf{r}}r\cos\phi+\hat{\boldsymbol{\phi}}\sin\phi)\\
&= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial}{\partial\phi}0-\frac{\partial}{\partial z}(\sin\phi)\right)+\hat{\boldsymbol{\phi}}\left(\frac{\partial}{\partial z}(r\cos\phi)-\frac{\partial}{\partial r}0\right)\\
&\quad+\hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}(r(\sin\phi))-\frac{\partial}{\partial\phi}(r\cos\phi)\right)\\
&= \hat{\mathbf{r}}0+\hat{\boldsymbol{\phi}}0+\hat{\mathbf{z}}\frac{1}{r}(\sin\phi+(r\sin\phi))=\hat{\mathbf{z}}\sin\phi\left(1+\frac{1}{r}\right),\\
\iint\nabla\times\mathbf{B}\cdot d\mathbf{s} &= \int_{\phi=0}^{\pi}\int_{r=0}^{2}\left(\hat{\mathbf{z}}\sin\phi\left(1+\frac{1}{r}\right)\right)\cdot(\hat{\mathbf{z}}r\,dr\,d\phi)\\
&= \int_{\phi=0}^{\pi}\int_{r=0}^{2}\sin\phi(r+1)\,dr\,d\phi=\left(\left(-\cos\phi\left(\tfrac{1}{2}r^2+r\right)\right)\Big|_{r=0}^{2}\right)\Big|_{\phi=0}^{\pi}=8.
\end{aligned}$$

Problem 3.46 Repeat Problem 3.45 for the contour shown in Fig. P3.45(b).

Solution: (a)

$$\oint \mathbf{B}\cdot d\mathbf{l} = \int_{L_1} \mathbf{B}\cdot d\mathbf{l} + \int_{L_2} \mathbf{B}\cdot d\mathbf{l} + \int_{L_3} \mathbf{B}\cdot d\mathbf{l} + \int_{L_4} \mathbf{B}\cdot d\mathbf{l},$$

$$\mathbf{B}\cdot d\mathbf{l} = (\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)\cdot(\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}} r\,d\phi + \hat{\mathbf{z}}\,dz) = r\cos\phi\,dr + r\sin\phi\,d\phi,$$

$$\int_{L_1} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=1}^{2} r\cos\phi\,dr\right)\bigg|_{\phi=0,\ z=0} + \left(\int_{\phi=0}^{0} r\sin\phi\,d\phi\right)\bigg|_{z=0}$$

$$= \left(\tfrac{1}{2}r^2\right)\big|_{r=1}^{2} + 0 = \frac{3}{2},$$

$$\int_{L_2} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{2} r\cos\phi\,dr\right)\bigg|_{z=0} + \left(\int_{\phi=0}^{\pi/} r\sin\phi\,d\phi\right)\bigg|_{r=2,\ z=0}$$

$$= 0 + (-2\cos\phi)\big|_{\phi=0}^{\pi/2} = 2,$$

$$\int_{L_3} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{1} r\cos\phi\,dr\right)\bigg|_{\phi=\pi/2,\ z=0} + \left(\int_{\phi=\pi/2}^{\pi/2} r\sin\phi\,d\phi\right)\bigg|_{z=0} = 0,$$

$$\int_{L_4} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=1}^{1} r\cos\phi\,dr\right)\bigg|_{z=0} + \left(\int_{\phi=\pi/2}^{0} r\sin\phi\,d\phi\right)\bigg|_{r=1,\ z=0}$$

$$= 0 + (-\cos\phi)\big|_{\phi=\pi/2}^{0} = -1,$$

$$\oint \mathbf{B}\cdot d\mathbf{l} = \frac{3}{2} + 2 + 0 - 1 = \frac{5}{2}.$$

(b)

$$\nabla\times\mathbf{B} = \nabla\times(\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)$$

$$= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial}{\partial\phi}0 - \frac{\partial}{\partial z}(\sin\phi)\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial}{\partial z}(r\cos\phi) - \frac{\partial}{\partial r}0\right)$$

$$+ \hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}(r(\sin\phi)) - \frac{\partial}{\partial\phi}(r\cos\phi)\right)$$

$$= \hat{\mathbf{r}}0 + \hat{\boldsymbol{\phi}}0 + \hat{\mathbf{z}}\frac{1}{r}(\sin\phi + (r\sin\phi)) = \hat{\mathbf{z}}\sin\phi\left(1 + \frac{1}{r}\right),$$

$$\iint \nabla\times\mathbf{B}\cdot d\mathbf{s} = \int_{\phi=0}^{\pi/2}\int_{r=1}^{2}\left(\hat{\mathbf{z}}\sin\phi\left(1+\frac{1}{r}\right)\right)\cdot(\hat{\mathbf{z}} r\,dr\,d\phi)$$

$$= \int_{\phi=0}^{\pi/2}\int_{r=1}^{2} \sin\phi\,(r+1)\,dr\,d\phi$$

$$= \left(\left(-\cos\phi\left(\tfrac{1}{2}r^2 + r\right)\right)\big|_{r=1}^{2}\right)\bigg|_{\phi=0}^{\pi/2} = \frac{5}{2}.$$