Problem 4.20 Given the electric flux density

$$\mathbf{D} = \hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y) \quad (\text{C/m}^2)$$

determine

(a) ρ_v by applying Eq. (4.26).

(b) The total charge Q enclosed in a cube 2 m on a side, located in the first octant with three of its sides coincident with the x-, y-, and z-axes and one of its corners at the origin.

(c) The total charge Q in the cube, obtained by applying Eq. (4.29).

Solution:

(a) By applying Eq. (4.26)

$$\rho_\text{v} = \nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}(2x+2y) + \frac{\partial}{\partial y}(3x-2y) = 0.$$

(b) Integrate the charge density over the volume as in Eq. (4.27):

$$Q = \int_{\mathscr{V}} \nabla \cdot \mathbf{D} \, d\mathscr{V} = \int_{x=0}^{2} \int_{y=0}^{2} \int_{z=0}^{2} 0 \, dx \, dy \, dz = 0.$$

(c) Apply Gauss' law to calculate the total charge from Eq. (4.29)

$$Q = \oint \mathbf{D} \cdot d\mathbf{s} = F_\text{front} + F_\text{back} + F_\text{right} + F_\text{left} + F_\text{top} + F_\text{bottom},$$

$$F_\text{front} = \int_{y=0}^{2} \int_{z=0}^{2} \left(\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y)\right)\Big|_{x=2} \cdot (\hat{\mathbf{x}} \, dz \, dy)$$

$$= \int_{y=0}^{2} \int_{z=0}^{2} 2(x+y)\Big|_{x=2} \, dz \, dy = \left(2z \left(2y + \frac{1}{2}y^2 \right) \Bigg|_{z=0}^{2} \right) \Bigg|_{y=0}^{2} = 24,$$

$$F_\text{back} = \int_{y=0}^{2} \int_{z=0}^{2} \left(\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y)\right)\Big|_{x=0} \cdot (-\hat{\mathbf{x}} \, dz \, dy)$$

$$= -\int_{y=0}^{2} \int_{z=0}^{2} 2(x+y)\Big|_{x=0} \, dz \, dy = -\left(zy^2 \Big|_{z=0}^{2} \right) \Bigg|_{y=0}^{2} = -8,$$

$$F_\text{right} = \int_{x=0}^{2} \int_{z=0}^{2} \left(\hat{\mathbf{x}}2(x+y) + \hat{\mathbf{y}}(3x-2y)\right)\Big|_{y=2} \cdot (\hat{\mathbf{y}} \, dz \, dx)$$

$$= \int_{x=0}^{2} \int_{z=0}^{2} (3x-2y)\Big|_{y=2} \, dz \, dx = \left(z \left(\frac{3}{2}x^2 - 4x \right) \Bigg|_{z=0}^{2} \right) \Bigg|_{x=0}^{2} = -4,$$

$$F_{\text{left}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y)+\hat{\mathbf{y}}(3x-2y))\Big|_{y=0} \cdot (-\hat{\mathbf{y}}\,dz\,dx)$$

$$= -\int_{x=0}^{2}\int_{z=0}^{2} (3x-2y)\Big|_{y=0}\,dz\,dx = -\left(z\left(\frac{3}{2}x^2\right)\Big|_{z=0}^{2} \right)\Big|_{x=0}^{2} = -12,$$

$$F_{\text{top}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y)+\hat{\mathbf{y}}(3x-2y))\Big|_{z=2} \cdot (\hat{\mathbf{z}}\,dy\,dx)$$

$$= \int_{x=0}^{2}\int_{z=0}^{2} 0\Big|_{z=2}\,dy\,dx = 0,$$

$$F_{\text{bottom}} = \int_{x=0}^{2}\int_{z=0}^{2} (\hat{\mathbf{x}}2(x+y)+\hat{\mathbf{y}}(3x-2y))\Big|_{z=0} \cdot (\hat{\mathbf{z}}\,dy\,dx)$$

$$= \int_{x=0}^{2}\int_{z=0}^{2} 0\Big|_{z=0}\,dy\,dx = 0.$$

Thus $Q = \oint \mathbf{D}\cdot d\mathbf{s} = 24-8-4-12+0+0 = 0.$

Problem 4.22 Charge Q_1 is uniformly distributed over a thin spherical shell of radius a, and charge Q_2 is uniformly distributed over a second spherical shell of radius b, with $b > a$. Apply Gauss's law to find $\mathbf{E}$ in the regions $R < a$, $a < R < b$, and $R > b$.

Solution: Using symmetry considerations, we know $\mathbf{D} = \hat{\mathbf{R}} D_\mathrm{R}$. From Table 3.1, $d\mathbf{s} = \hat{\mathbf{R}} R^2 \sin\theta \, d\theta \, d\phi$ for an element of a spherical surface. Using Gauss's law in integral form (Eq. (4.29)),

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q_\mathrm{tot},$$

where Q_tot is the total charge enclosed in S. For a spherical surface of radius R,

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (\hat{\mathbf{R}} D_\mathrm{R}) \cdot (\hat{\mathbf{R}} R^2 \sin\theta \, d\theta \, d\phi) = Q_\mathrm{tot},$$

$$D_\mathrm{R} R^2 (2\pi) [-\cos\theta]_0^\pi = Q_\mathrm{tot},$$

$$D_\mathrm{R} = \frac{Q_\mathrm{tot}}{4\pi R^2} .$$

From Eq. (4.15), we know a linear, isotropic material has the constitutive relationship $\mathbf{D} = \varepsilon \mathbf{E}$. Thus, we find $\mathbf{E}$ from $\mathbf{D}$.

(a) In the region $R < a$,

$$Q_\mathrm{tot} = 0, \qquad \mathbf{E} = \hat{\mathbf{R}} E_\mathrm{R} = \frac{\hat{\mathbf{R}} Q_\mathrm{tot}}{4\pi R^2 \varepsilon} = 0 \quad \text{(V/m)}.$$

(b) In the region $a < R < b$,

$$Q_\mathrm{tot} = Q_1, \qquad \mathbf{E} = \hat{\mathbf{R}} E_\mathrm{R} = \frac{\hat{\mathbf{R}} Q_1}{4\pi R^2 \varepsilon} \quad \text{(V/m)}.$$

(c) In the region $R > b$,

$$Q_\mathrm{tot} = Q_1 + Q_2, \qquad \mathbf{E} = \hat{\mathbf{R}} E_\mathrm{R} = \frac{\hat{\mathbf{R}} (Q_1 + Q_2)}{4\pi R^2 \varepsilon} \quad \text{(V/m)}.$$

Problem 4.23 The electric flux density inside a dielectric sphere of radius a centered at the origin is given by

$$\mathbf{D} = \hat{\mathbf{R}}\rho_0 R \qquad (\text{C/m}^2)$$

where ρ_0 is a constant. Find the total charge inside the sphere.

Solution:

$$\begin{aligned} Q &= \oint_S \mathbf{D} \cdot d\mathbf{s} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \hat{\mathbf{R}}\rho_0 R \cdot \hat{\mathbf{R}} R^2 \sin\theta \, d\theta \, d\phi \bigg|_{R=a} \\ &= 2\pi\rho_0 a^3 \int_0^{\pi} \sin\theta \, d\theta = -2\pi\rho_0 a^3 \cos\theta \big|_0^{\pi} = 4\pi\rho_0 a^3 \quad \text{(C)}. \end{aligned}$$