Units and dimensions

• Six fundamental International System of Units

Dimensions	Unit	symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric Current	Ampere	A
Temperature	Kelvin	K
Amount of substance	mole	mol

• any other dimension can be derived from the fundamental dimensions, e.g.:

$$F = ma = m\frac{dv}{dt} = kgm/s^2$$

$$E = \frac{F}{q} = \frac{F}{\int Idt} = kgm/As^3$$

Electromagnetic spectrum

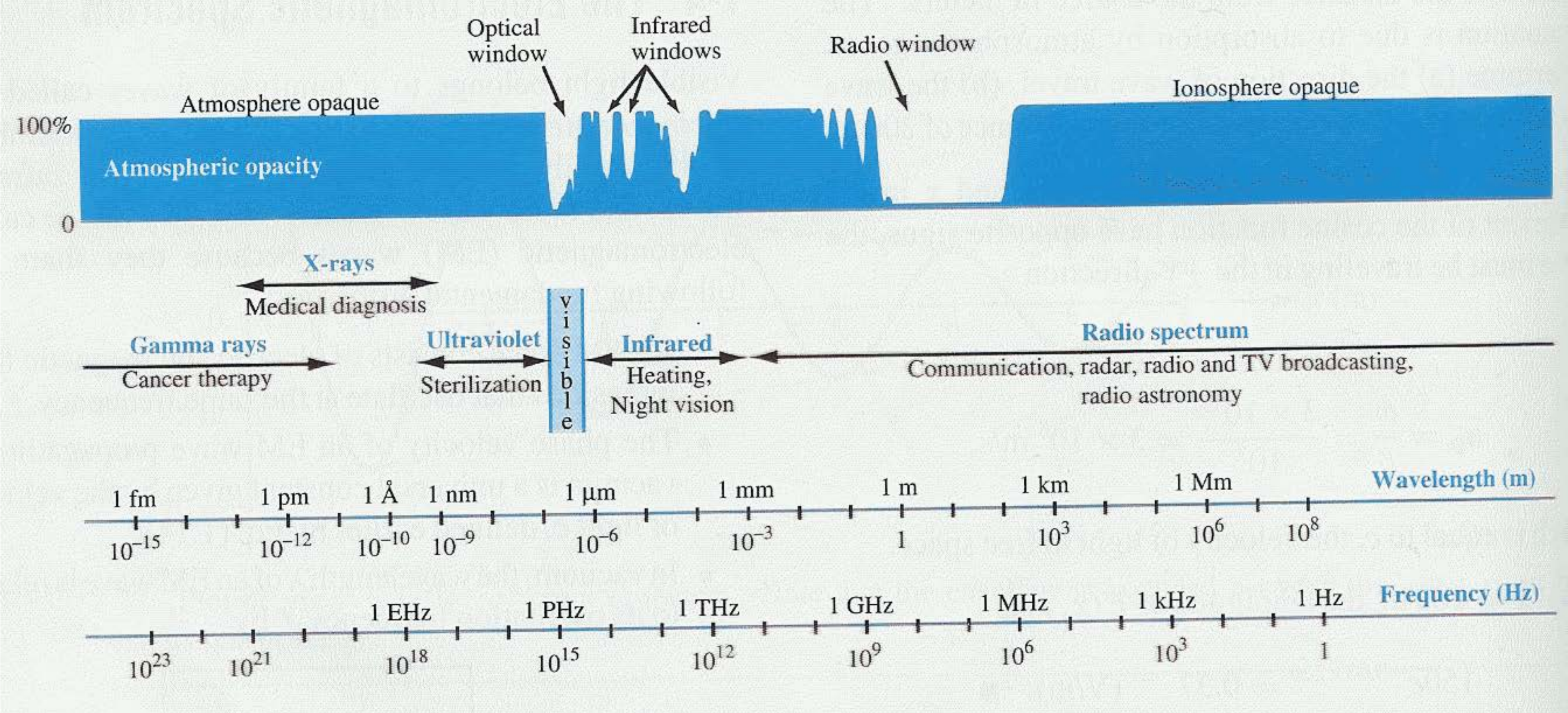


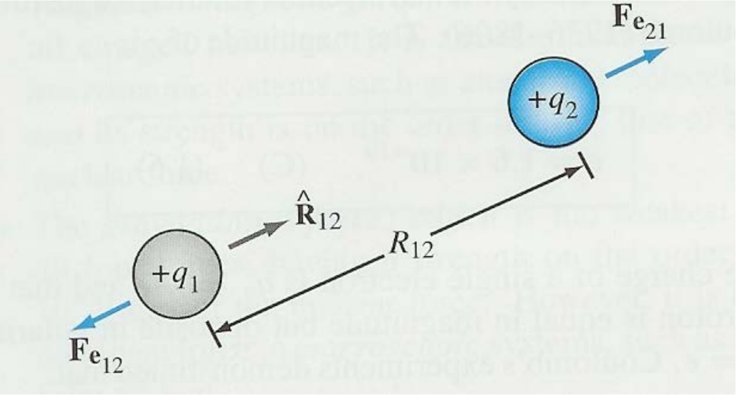
Figure 1-15: The electromagnetic spectrum.

Electromagnetic bands and applications

Frequency (Hz)	Band	Applications
10^{12}		
300 GHz (Microwave)	Extremely High Frequency EHF (30 - 300 GHz)	Radar, advanced communication systems, remote sensing, radio astronomy
(Microwave)	Super High Frequency SHF (3 - 30 GHz)	Radar, satellite communication systems, aircraft navigation, radio astronomy, remote sensing
1 GHz — 10^{9} (Microwave)	Ultra High Frequency UHF (300 MHz - 3 GHz)	TV broadcasting, radar, radio astronomy, microwave ovens, cellular telephone
(Microwave)	Very High Frequency VHF (30 - 300 MHz)	TV and FM broadcasting, mobile radio communication, air traffic control
	High Frequency HF (3 - 30 MHz)	Short wave broadcasting
1 MHz — 10^{6}	Medium Frequency MF (300 kHz - 3 MHz)	AM broadcasting
	Low Frequency LF (30 - 300 kHz)	Radio beacons, weather broadcast stations for air navigation
	Very Low Frequency VLF (3 - 30 kHz)	Navigation and position location
1 kHz — 10^{3}	Ultra Low Frequency ULF (300 Hz - 3 kHz)	Audio signals on telephone
	Super Low Frequency SLF (30 - 300 Hz)	Ionospheric sensing, electric power distribution, submarine communication
	Extremely Low Frequency ELF (3 - 30 Hz)	Detection of buried metal objects
1 Hz	$f < 3$ Hz)	Magnetotelluric sensing of the earth's structure

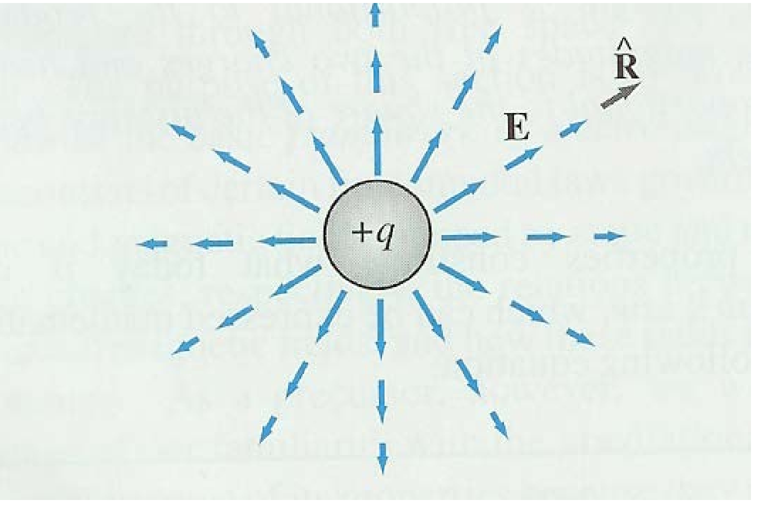
Electric field

- Electric forces on point charges, Columb's law



$$\vec{F}_{12} = \hat{R}_{12} \frac{q_1 q_2}{4\pi\varepsilon_0 R_{12}^2},$$

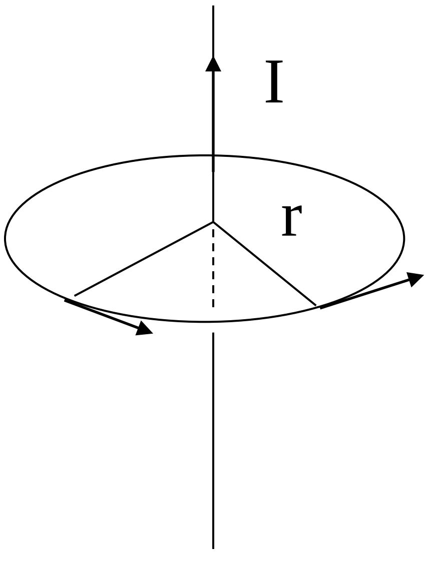
$$\vec{F}_{21} = \hat{R}_{21} \frac{q_1 q_2}{4\pi\varepsilon_0 R_{21}^2},$$



$$\vec{F}_{21} = q_1 \frac{q}{4\pi\varepsilon_0 R^2} = q_1\vec{E},$$

$$\vec{E} = \hat{R} \frac{q}{4\pi\varepsilon_0 R^2},$$

Magnetic field by constant current



$$B = \frac{\mu I}{2\pi r}\hat{\phi}$$

$\mu = \mu_r \mu_0$,

μ_r: relative magnetic permeability

$\mu_r = 1$ for most materials

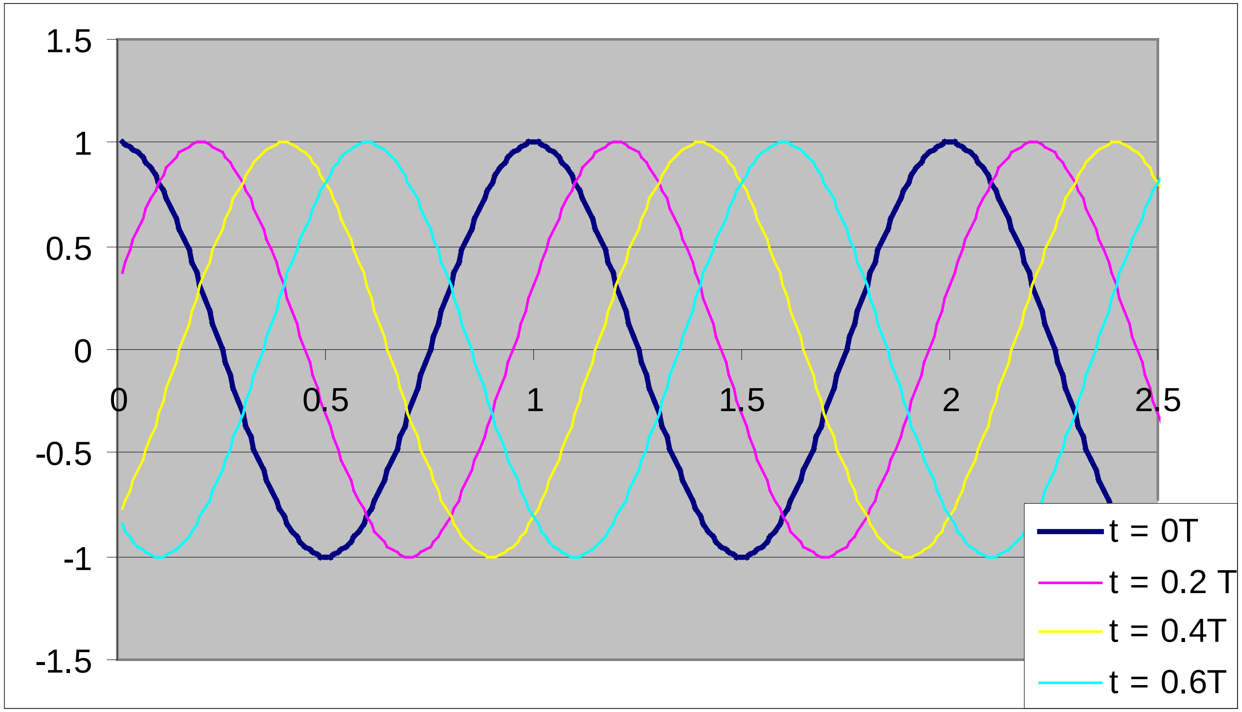
$$H = \frac{B}{\mu}$$

$$= \frac{I}{2\pi r}\hat{\phi}$$

Traveling wave

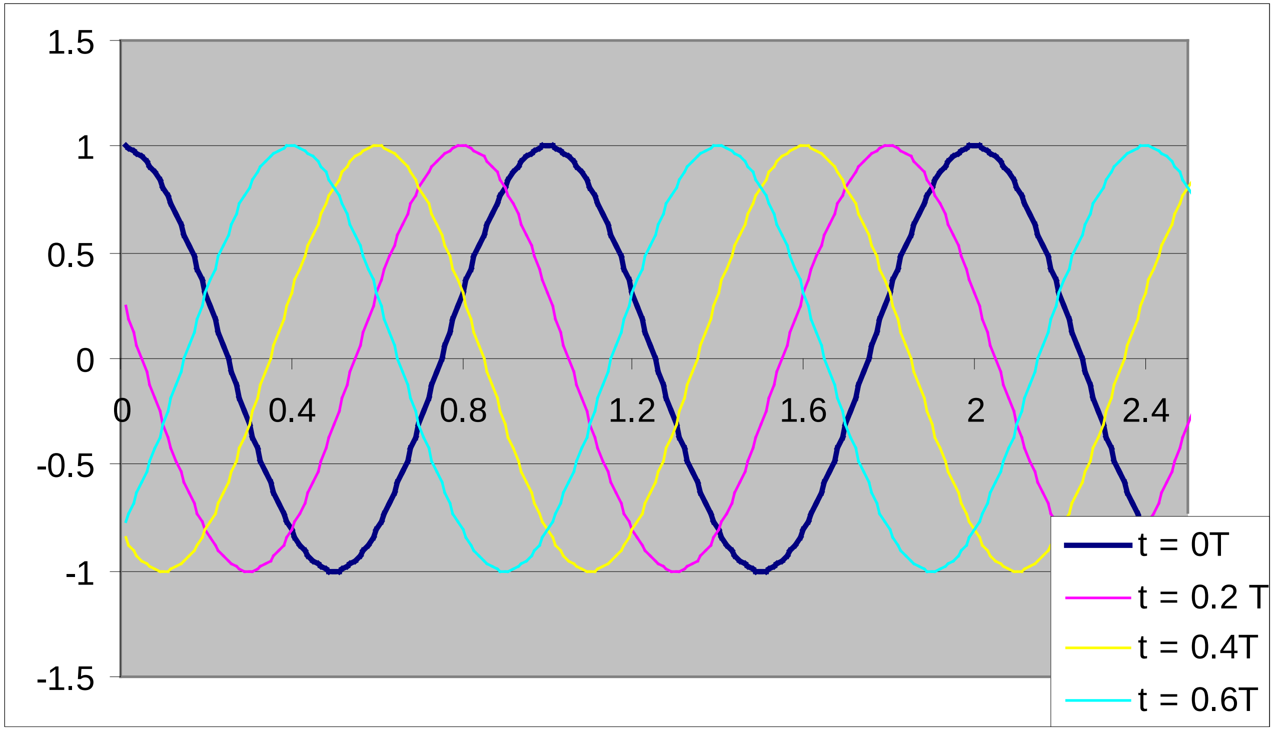
$$y(x,t) = A\cos(2\pi t/T - 2\pi x/\lambda),$$

$$\phi(x,t) = 2\pi t/T - 2\pi x/\lambda, \qquad y(x,t) = A\cos\phi(x,t),$$



Traveling wave

$$y(x,t) = A\cos(2\pi t/T + 2\pi x/\lambda),$$



Velocity = 0.6λ/0.6T

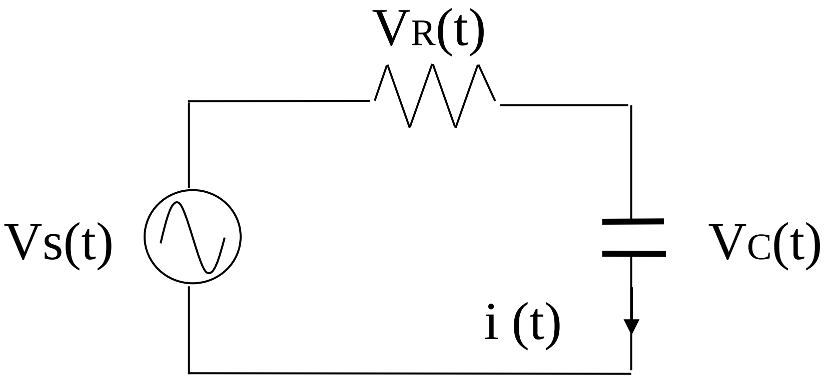
= λ/T

Phase velocity:

Vp = dx/dt

= - λ/T

- Phasor



$V_s(t) = V_0 Sin(\omega t + \phi_0)$,

$V_R(t) = i(t)R$,

$V_C(t) = \int i(t)dt/C$,

$V_s(t) = V_R(t) + V_C(t)$,

⇒ $V_0 Sin(\omega t + \phi_0) = \int i(t)dt/C + i(t)R$, Integral equation,

Using phasor to solve integral and differential equations

- Phasor

$$Z(t) = \mathrm{Re}(\tilde{Z}\, e^{j\omega t})$$

$\tilde{Z}$ is time independent function of Z(t), i.e. phasor

$$V_s(t) = V_0 \mathrm{Sin}(\omega t + \phi_0)$$

$$= \mathrm{Re}(V_0\, e^{j(\phi_0 - \pi/2)}\; e^{j\omega t})$$

$$= \mathrm{Re}(\tilde{V}\; e^{j\omega t}),$$

$$\tilde{V} = V_0\, e^{j(\phi_0 - \pi/2)},$$

- Phasor

$$i(t) = \mathrm{Re}(\tilde{I}\, e^{j\omega t})$$

$$\int i(t)dt = \int \mathrm{Re}(\tilde{I}\, e^{j\omega t})dt$$

$$= \mathrm{Re}(\tilde{I}\, \frac{1}{j\omega}\, e^{j\omega t}),$$

<u>time domain equation:</u>

$$V_0 \mathrm{Sin}(\omega t+\phi_0) = \int i(t)dt/C + i(t)R,$$ ⇒

<u>phasor domain equation:</u>

$$\mathrm{Re}(\tilde{V}\, e^{j\omega t}) = \mathrm{Re}(\tilde{I}\, \frac{1}{j\omega}\, e^{j\omega t})/C + \mathrm{Re}(\tilde{I}\, e^{j\omega t}),$$

• <u>Phasor domain</u>

$$\tilde{V} = \tilde{I}\,\frac{1}{j\omega C} + \tilde{I}\,R,$$

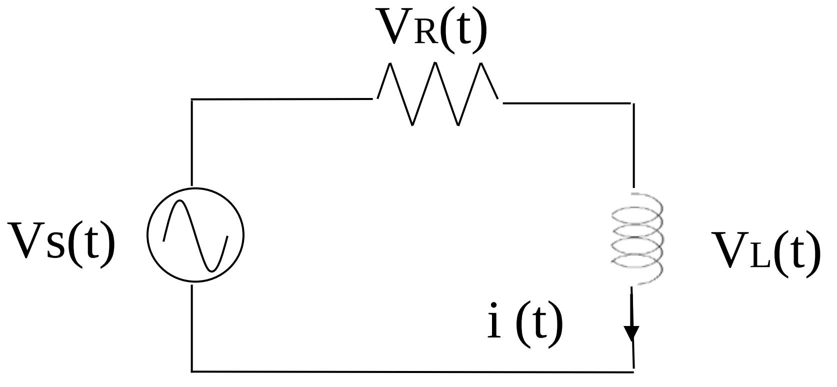
$$\tilde{I} = \frac{\tilde{V}}{R + 1/(j\omega C)}$$

$$= \frac{V_0\, e^{j(\phi_0 - \pi/2)}}{R + 1/(j\omega C)} \;,$$

<u>Back to time domain:</u>

$$i(t) = \mathrm{Re}(\tilde{I}\, e^{j\omega t}) = \mathrm{Re}\left(\frac{V_0\, e^{j(\phi_0 - \pi/2)}}{R + 1/(j\omega C)}\, e^{j\omega t}\right)$$

• An Example :



$V_s(t) = V_0 \sin(\omega t + \phi_0)$,

$V_R(t) = i(t)R$,

$V_L(t) = L\,di(t)/dt$,

$V_s(t) = V_R(t) + V_L(t)$,

⇨ $V_0 \sin(\omega t + \phi_0) = L\,di(t)/dt + i(t)R$, differential equation,

Using phasor to solve the differential equation.

- Phasor

$$i(t) = \mathrm{Re}(\tilde{I}\, e^{j\omega t})$$

$$di(t)/dt = \mathrm{Re}(d\tilde{I}\, e^{j\omega t})/dt$$

$$= \mathrm{Re}(\tilde{I}\, j\omega e^{j\omega t}),$$

time domain equation:

$$V_0 \mathrm{Sin}(\omega t+\phi_0) = L di(t)/dt + i(t)R,$$

phasor domain equation:

$$\mathrm{Re}(\tilde{V}\, e^{j\omega t}) = \mathrm{Re}(\tilde{I}\, j\omega\, e^{j\omega t})L + \mathrm{Re}(\tilde{I}\, e^{j\omega t}),$$

• Phasor domain

$$\tilde{V} = \tilde{I}\, j\omega L + \tilde{I} R,$$

$$\tilde{I} = \frac{\tilde{V}}{R + (j\omega L)}$$

$$= \frac{V_0 e^{j(\phi_0 - \pi/2)}}{R + j\omega L)},$$

Back to time domain:

$$i(t) = \mathrm{Re}(\tilde{I} e^{j\omega t}) = \mathrm{Re}\left(\frac{V_0 e^{j(\phi_0 - \pi/2)}}{R + (j\omega L)} e^{j\omega t} \right)$$

- Steps of transferring integral or differential equations to linear equations using phasor.

1. Express time-dependent variables as phsaor.
2. Rewrite integral or differential equations in phasor domain.
3. Solve phasor domain equations
4. Change phasors variable to their time domain value

16.360 Lecture 3

- Electromagnetic spectrum.

  Recall relation: $\lambda f = v$.

- Some important wavelength ranges:

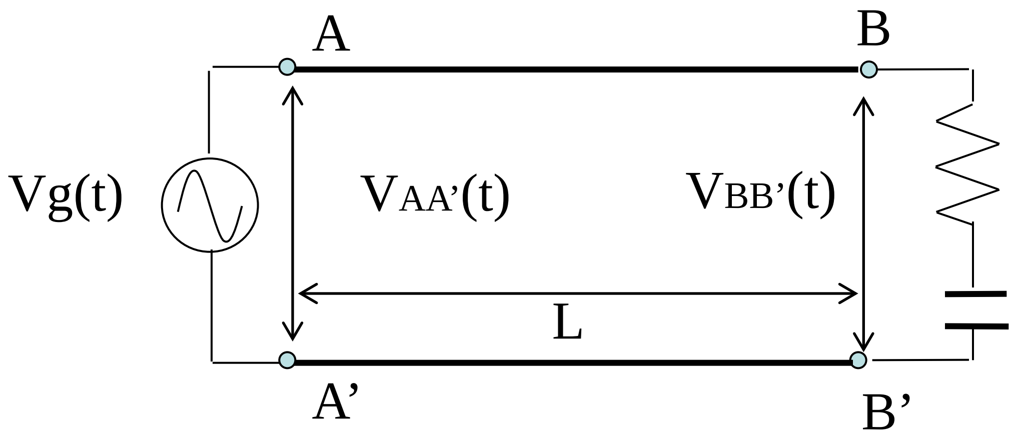
  1. Fiber optical communication: λ = 1.3 – 1.5μm.
  2. Free space communication: ~ 700nm – 980nm.
  3. TV broadcasting and cellular phone: 300MHz – 3GHz.
  4. Radar and remote sensing: 30GHz – 300GHz

16.360 Lecture 3

- Transmission lines

1. Transmission line parameters, equations
2. Wave propagations
3. Lossless line, standing wave and reflection coefficient
4. Input impedence
5. Special cases of lossless line
6. Power flow
7. Smith chart
8. Impedence matching
9. Transients on transmission lines

1. Transmission line parameters, equations



$V_{AA'}(t) = Vg(t) = V0\cos(\omega t)$,

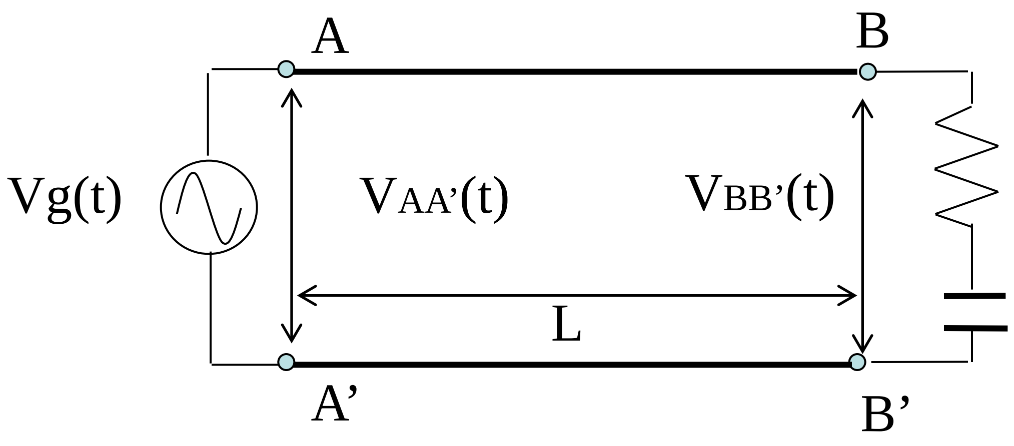
Low frequency circuits:

$V_{BB'}(t) = V_{AA'}(t)$ ← Approximate result

$V_{BB'}(t) = V_{AA'}(t-t_d) = V_{AA'}(t-L/c)$

$= V0\cos(\omega(t-L/c))$,

1. Transmission line parameters, equations



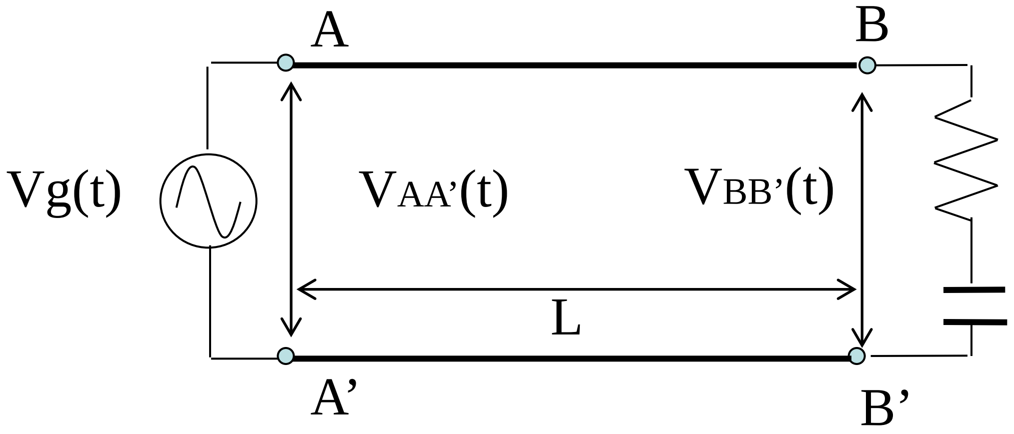
Recall: $f\lambda = c$, and $\omega = 2\pi f$

$$V_{BB'}(t) = V_{AA'}(t - t_d) = V_{AA'}(t - L/c)$$
$$= V_0\cos(\omega(t - L/c))$$
$$= V_0\cos(\omega t - 2\pi L/\lambda),$$

If $\lambda >> L$, $V_{BB'}(t) \approx V_0\cos(\omega t) = V_{AA'}(t)$,

If $\lambda <= L$, $V_{BB'}(t) \neq V_{AA'}(t)$, the circuit theory has to be replaced.

- Transmission line parameters



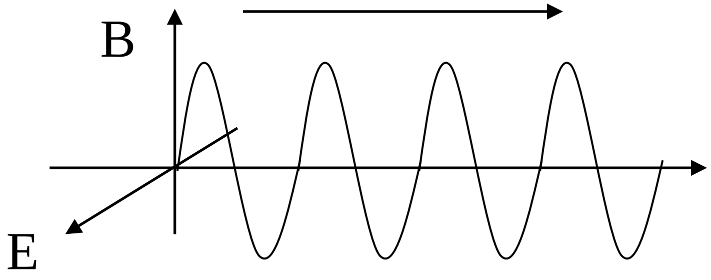
- time delay

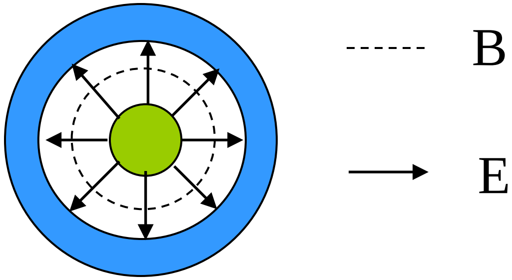
  $V_{BB'}(t) = V_{AA'}(t-t_d) = V_{AA'}(t-L/v_p)$,

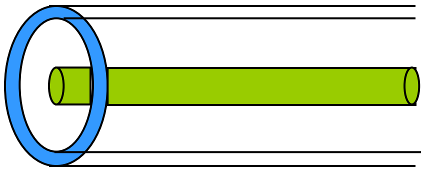
- Reflection: the voltage has to be treat as wave, some bounce back
- power loss: due to reflection and some other loss mechanism,
- Dispersion: in material, V_p could be different for different wavelength

16.360 Lecture 4

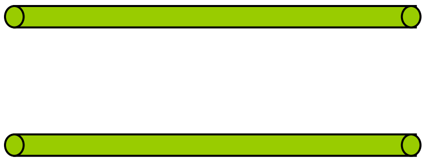
- Types of transmission lines
  - Transverse electromagnetic (TEM) transmission lines



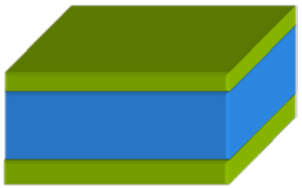




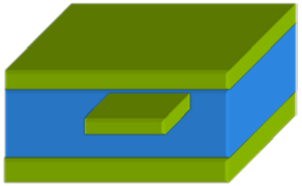
a) Coaxial line



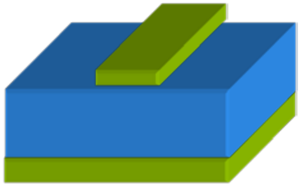
b) Two-wire line



c) Parallel-plate line



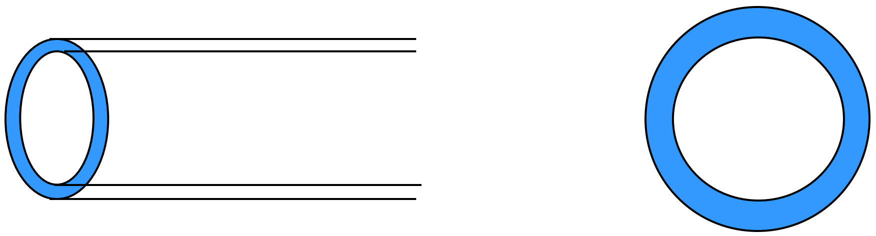
d) Strip line



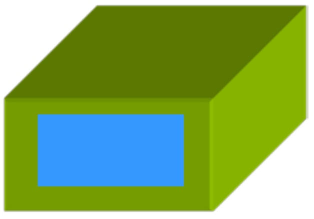
e) Microstrip line

16.360 Lecture 4

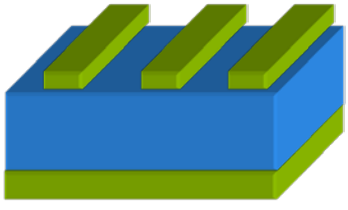
- Types of transmission lines
  - Higher-order transmission lines



a) Optical fiber

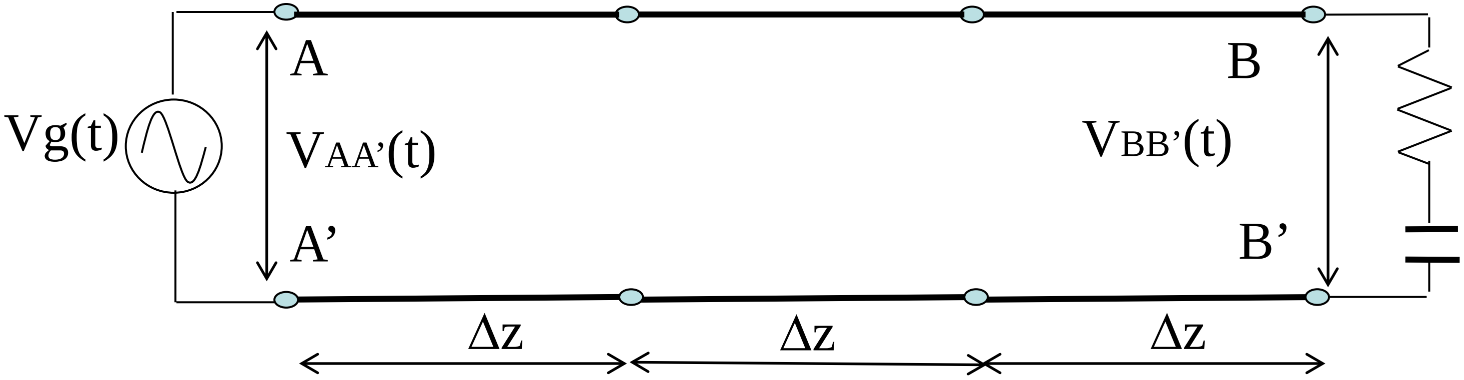


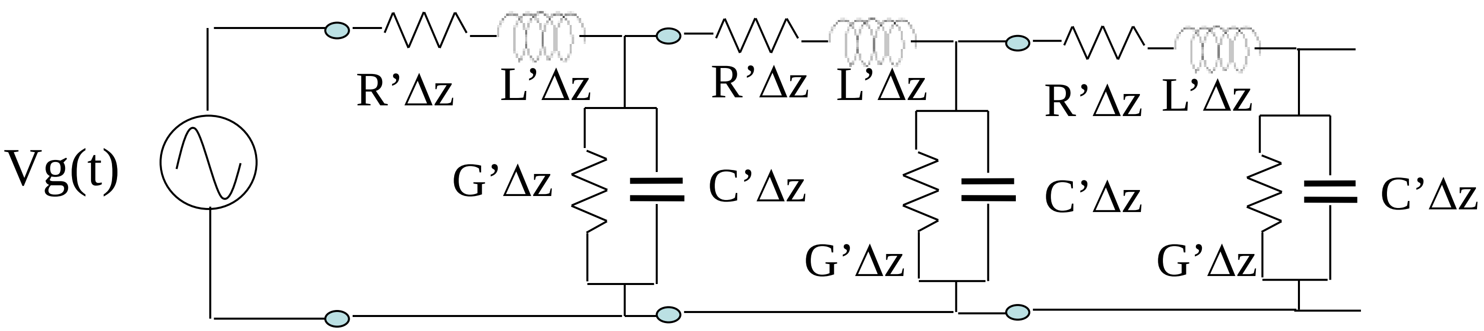
b) Rectangular waveguide



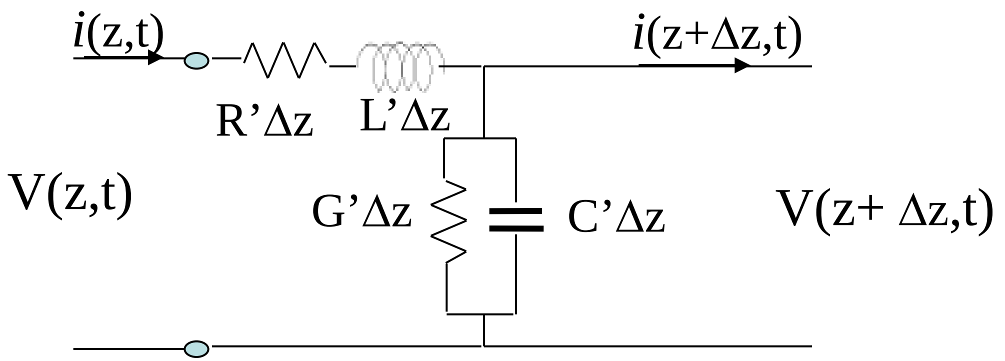
c) Coplanar waveguide

- Lumped-element Model
  - Represent transmission lines as parallel-wire configuration





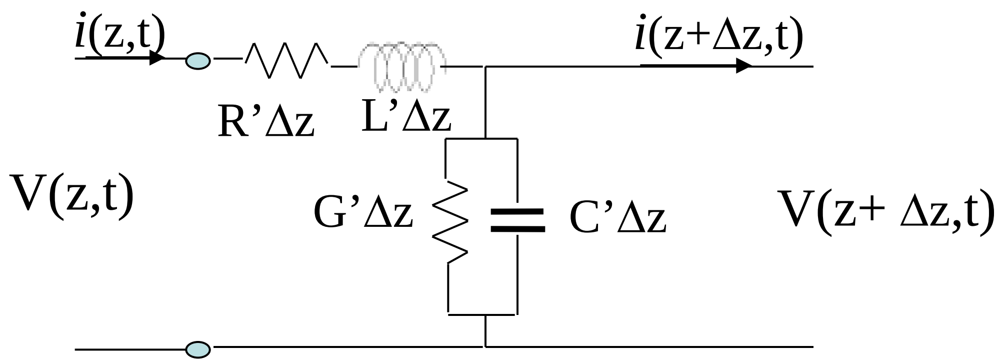
- Transmission line equations
  - Represent transmission lines as parallel-wire configuration



$$V(z,t) = R'\Delta z\, i(z,t) + L'\Delta z\, \partial i(z,t)/\partial t + V(z+\Delta z,t), \quad (1)$$

$$i(z,t) = G'\Delta z\, V(z+\Delta z,t) + C'\Delta z\, \partial V(z+\Delta z,t)/\partial t + i(z+\Delta z,t), \quad (2)$$

- Transmission line equations



$V(z,t) = R'\Delta z\, i(z,t) + L'\Delta z\, \partial i(z,t)/\partial t + V(z+\Delta z,t),$ (1)

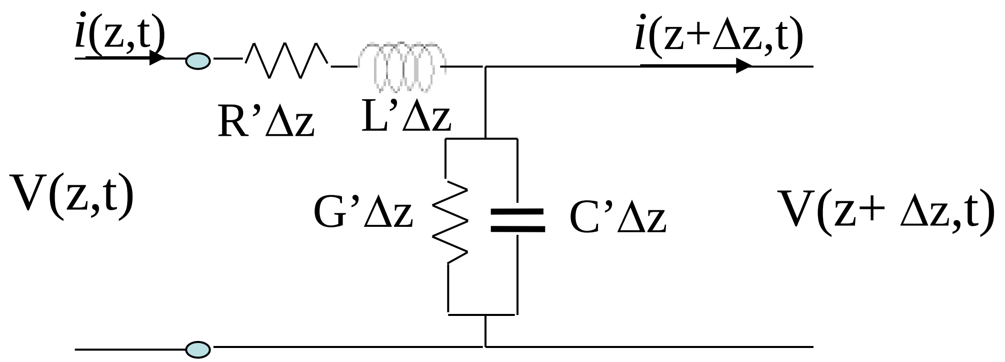
⇒ $-V(z+\Delta z,t) + V(z,t) = R'\Delta z\, i(z,t) + L'\Delta z\, \partial i(z,t)/\partial t$

⇒ $-\partial V(z,t)/\partial z = R'\, i(z,t) + L'\, \partial i(z,t)/\partial t,$ (3)

Rewrite V(z,t) and *i*(z,t) as phasors, for sinusoidal V(z,t) and *i*(z,t):

$V(z,t) = \mathrm{Re}(\tilde{V}(z)\, e^{j\omega t}),$ $\quad i(z,t) = \mathrm{Re}(\tilde{i}(z)\, e^{j\omega t}),$

- Transmission line equations



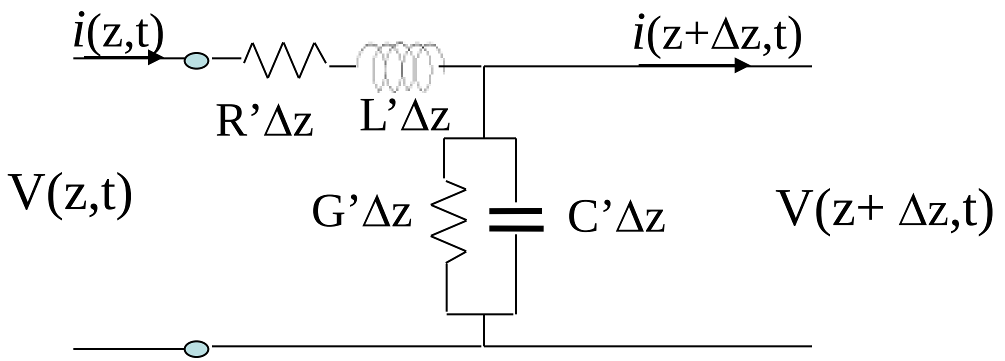
Recall:

$$d i(t)/dt = \mathrm{Re}(d\,\tilde{i}\, e^{j\omega t})/dt = \mathrm{Re}(\tilde{i}\, j\omega e^{j\omega t}),$$

$$-\partial V(z,t)/\partial z = R'\, i(z,t) + L'\, \partial\, i(z,t)/\partial t, \qquad (3)$$

$$\Rightarrow \quad -d\tilde{V}(z)/dz = R'\, \tilde{i}(z) + j\omega L'\, \tilde{i}(z), \qquad (4)$$

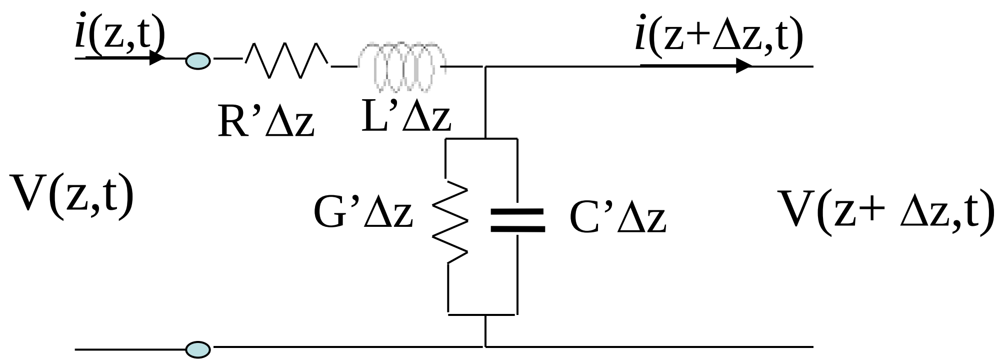
- Transmission line equations
  - Represent transmission lines as parallel-wire configuration



$$V(z,t) = R'\Delta z\, i(z,t) + L'\Delta z\, \partial i(z,t)/\partial t + V(z+\Delta z,t), \quad (1)$$

$$i(z,t) = G'\Delta z\, V(z+\Delta z,t) + C'\Delta z\, \partial V(z+\Delta z,t)/\partial t + i(z+\Delta z,t), \quad (2)$$

- Transmission line equations



$$i(z,t) = G'\Delta z\, V(z+\Delta z,t) + C'\Delta z\, \partial V(z+\Delta z,t)/\partial t + i(z+\Delta z,t), \qquad (2)$$

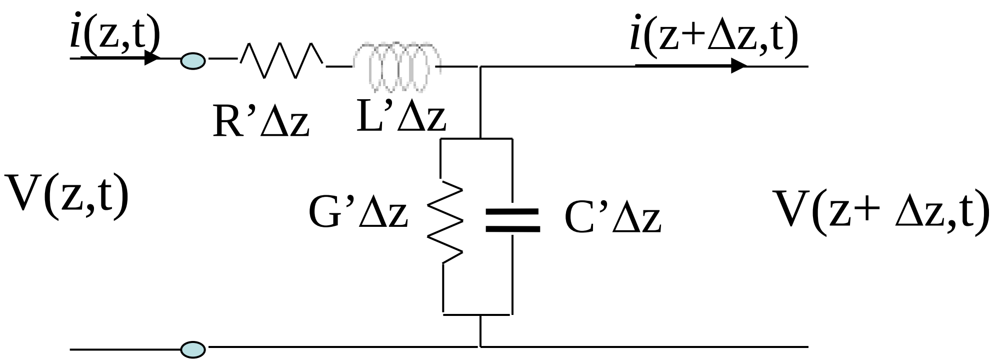
$$- i(z+\Delta z,t) + i(z,t) = G'\Delta z\, V(z+\Delta z,t) + C'\Delta z\, \partial V(z+\Delta z,t)/\partial t$$

$$- \partial i(z,t)/\partial z = G' V(z,t) + C' \partial V(z,t)/\partial t, \qquad (5)$$

Rewrite V(z,t) and i(z,t) as phasors, for sinusoidal V(z,t) and i(z,t):

$$V(z,t) = \mathrm{Re}(\tilde{V}(z)\, e^{j\omega t}), \qquad i(z,t) = \mathrm{Re}(\tilde{i}(z)\, e^{j\omega t}),$$

- Transmission line equations



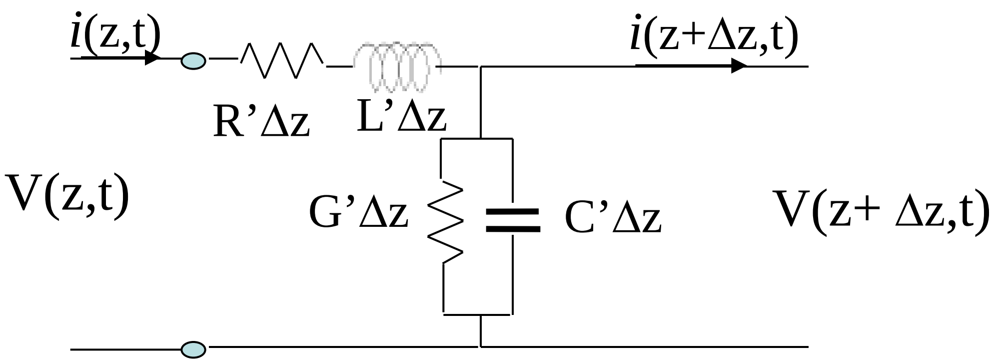
Recall:

$$dV(t)/dt = Re(d\,\tilde{V}\,e^{j\omega t})/dt \;=\; Re(\tilde{V} j\omega e^{j\omega t}),$$

$$-\,\partial i(z,t)/\partial z = G'\,V(z,t) + C'\,\partial\,V(z,t)/\,\partial t, \qquad (6)$$

$$-\,d\,\tilde{i}(z)/dz = G'\,\tilde{V}(z) + j\omega C'\,\tilde{V}(z), \qquad (7)$$

- Telegrapher's equation in phasor domain



$$
\begin{cases}
-\,d\tilde{V}(z)/dz = R'\,\tilde{i}(z) + j\omega L'\,\tilde{i}(z), & (4) \\
-\,d\,\tilde{i}(z)/dz = G'\,\tilde{V}(z) + j\omega C'\,\tilde{V}(z), & (7)
\end{cases}
$$

Take d /dz on both sides of eq. (4)

$$
\Rightarrow \quad -\,d^2\tilde{V}(z)/dz^2 = R'\,d\tilde{i}(z)/dz + j\omega L'\,d\tilde{i}(z)/dz, \qquad (8)
$$

- Telegrapher's equation in phasor domain

$$\begin{cases} -d\tilde{V}(z)/dz = R'\,\tilde{i}(z) + j\omega L'\,\tilde{i}(z), & (4) \\ -d\,\tilde{i}(z)/dz = G'\,\tilde{V}(z) + j\omega C'\,\tilde{V}(z), & (7) \end{cases}$$

$$-d^2\tilde{V}(z)/dz^2 = R'\,d\tilde{i}(z)/dz + j\omega L'\,d\tilde{i}(z)/dz, \quad (8)$$

substitute (7) to (8)

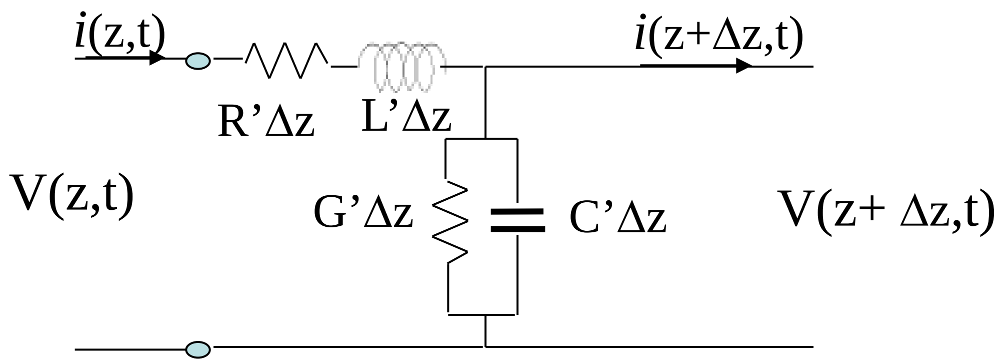
$$\Rightarrow \quad d^2\tilde{V}(z)/dz^2 = (R' + j\omega L')(G' + j\omega C')\tilde{V}(z), \quad \text{or}$$

$$d^2\tilde{V}(z)/dz^2 - (R' + j\omega L')(G' + j\omega C')\tilde{V}(z) = 0, \quad (9)$$

$$d^2\tilde{V}(z)/dz^2 - \gamma^2\tilde{V}(z) = 0, \quad (10)$$

$$\gamma^2 = (R' + j\omega L')(G' + j\omega C'), \quad (11)$$

- Telegrapher's equation in phasor domain



$$\begin{cases} -d\tilde{V}(z)/dz = R'\,\tilde{i}(z) + j\omega L'\,\tilde{i}(z), & (4) \\ -d\,\tilde{i}(z)/dz = G'\,\tilde{V}(z) + j\omega C'\,\tilde{V}(z), & (7) \end{cases}$$

Take d /dz on both sides of eq. (7)

$$\Rightarrow \quad -d^2\,\tilde{i}(z)/dz^2 = G'\,d\tilde{V}(z)/dz + j\omega C'\,d\tilde{V}(z)/dz, \qquad (12)$$

- Telegrapher's equation in phasor domain

$$\begin{cases} -\,d\tilde{V}(z)/dz = R'\,\tilde{i}(z) + j\omega L'\,\tilde{i}(z), & (4) \\ -\,d\,\tilde{i}(z)/dz = G'\,\tilde{V}(z) + j\omega C'\,\tilde{V}(z), & (7) \end{cases}$$

$$-\,d^2\,i(z)/dz^2 = G'\,dV(z)/dz + j\omega C'\,dV(z)/dz, \qquad (12)$$

substitute (4) to (12)

$$d^2\,\tilde{i}(z)/dz^2 = (R' + j\omega L')\,(G' + j\omega C')\,\tilde{i}(z), \qquad \text{or}$$

$$\boxed{d^2\,\tilde{i}(z)/dz^2 - (R' + j\omega L')\,(G' + j\omega C')\,\tilde{i}(z) = 0,} \qquad (9)$$

$$d^2\,\tilde{i}(z)/dz^2 - \gamma^2\tilde{i}(z) = 0, \qquad (13)$$

$$\boxed{\gamma^2 = (R' + j\omega L')\,(G' + j\omega C'),} \qquad (11)$$

- Wave equations

$$\begin{cases} d^2\tilde{V}(z)/dz^2 - \gamma^2\tilde{V}(z) = 0, & (10) \\ d^2\tilde{i}(z)/dz^2 - \gamma^2\tilde{i}(z) = 0, & (13) \end{cases}$$

$$\gamma = \alpha + j\beta,$$

$$\alpha = \mathrm{Re}\sqrt{(R' + j\omega L')(G' + j\omega C')}\,,$$

$$\beta = \mathrm{Im}\sqrt{(R' + j\omega L')(G' + j\omega C')}\,,$$

16.360 Lecture 4

- Summary
  - Transmission line parameters
  - Types of transmission lines
  - Lumped-element model
  - Transmission line equations
  - Telegrapher's equations
  - Wave equations

16.360 Lecture 4

- Next lecture
  - Wave propagation on a Transmission line
  - Characteristic impedance
  - Standing wave and traveling wave
  - Lossless transmission line
  - Reflection coefficient

16.360 Lecture 5

- Today:
  - Wave propagation on a Transmission line
  - Characteristic impedance
  - Standing wave and traveling wave
  - Lossless transmission line
  - Reflection coefficient

- Wave equations

$$\begin{cases} d^2\tilde{V}(z)/dz^2 - \gamma^2\tilde{V}(z) = 0, & (10) \\ d^2\tilde{i}(z)/dz^2 - \gamma^2\tilde{i}(z) = 0, & (13) \end{cases}$$

$$\gamma = \alpha + j\beta,$$

$$\alpha = \mathrm{Re}\sqrt{(R' + j\omega L')(G' + j\omega C')},$$

$$\beta = \mathrm{Im}\sqrt{(R' + j\omega L')(G' + j\omega C')},$$

Solving the second order differential equation

$$\begin{cases} \tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} & (14) \\ \tilde{i}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} & (15) \end{cases}$$

- Wave equations

$$\begin{cases} \tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} & (14) \\ \tilde{i}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} & (15) \end{cases}$$

<u>where:</u>

V_0^+ and V_0^- are determined by boundary conditions.

I_0^+ and I_0^- are related to V_0^+ and V_0^- by characteristic impedance Z_0.

- Characteristic impedance Z_0

$$\begin{cases} \tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} & (14) \\ \tilde{i}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} & (15) \end{cases}$$

<u>recall:</u>

$$- d\tilde{V}(z)/dz = R' \tilde{i}(z) + j\omega L' \tilde{i}(z), \quad (4)$$

$$\Rightarrow \quad \gamma V_0^+ e^{-\gamma z} - \gamma V_0^- e^{\gamma z} = (R' + j\omega L') \tilde{i}(z), \quad (16)$$

$$\Rightarrow \quad \tilde{i}(z) = \frac{\gamma}{(R' + j\omega L')} (V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z})$$

$$\Rightarrow \quad I_0^+ = \frac{\gamma}{(R' + j\omega L')} V_0^+ \quad (17) \qquad I_0^- = \frac{-\gamma}{(R' + j\omega L')} V_0^- \quad (18)$$

- Characteristic impedance Z_0

$$I_0^+ = \frac{\gamma}{(R' + j\omega L')} V_0^+ \qquad (17)$$

$$I_0^- = \frac{-\gamma}{(R' + j\omega L')} V_0^- \qquad (18)$$

Define characteristic impedance Z_0

recall:

$$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}$$

$$Z_0 \equiv \frac{V_0^+}{I_0^+} = \frac{(R' + j\omega L')}{\gamma}$$

$$= \sqrt{\frac{(R' + j\omega L')}{(G' + j\omega C')}}$$

- Summary:

$$\begin{cases} \tilde{V}(z) = V_0^{+} e^{-\gamma z} + V_0^{-} e^{\gamma z} & (14) \\ \tilde{i}(z) = I_0^{+} e^{-\gamma z} + I_0^{-} e^{\gamma z} & (15) \end{cases}$$

$$\gamma = \sqrt{(R' + j\omega L')\,(G' + j\omega C')} \qquad (19)$$

$$Z_0 \equiv \frac{V_0^{+}}{I_0^{+}} = \sqrt{\frac{(R' + j\omega L')}{(G' + j\omega C')}} \qquad (20)$$

- Example, an air line :

$R' = 0\ \Omega$, $G' = 0\ /\Omega$, $Z_0 = 50\Omega$, $\beta = 20$ rad/m, f = 700 MHz

$L' = ?$ and $C' = ?$

solution:

$$Z_0 = \sqrt{\frac{(R' + j\omega L')}{(G' + j\,\omega C')}} = \sqrt{\frac{j\omega L'}{j\,\omega C'}} = 50\Omega$$

$$\gamma = \sqrt{(R' + j\omega L')\,(G' + j\omega C')} = j\omega\sqrt{L'C'}$$

$$\gamma = \alpha + j\beta,$$

$$\beta = \omega\sqrt{L'C'} = 20 \text{ rad/m}$$

- lossless transmission line :

$\gamma = \alpha + j\beta,$

$= \sqrt{(R' + j\omega L')\,(G' + j\omega C')}$

If R'<< j ωL' and G' << jωC',

$\gamma = \sqrt{(R' + j\,\omega L')\,(G' + j\omega C')}$

$= j\omega\sqrt{L'C'}$

$\alpha = 0$

$\beta = \omega\,\sqrt{L'C'}$

lossless line

- <u>lossless transmission line :</u>

$$Z_0 = \sqrt{\frac{(R' + j\omega L')}{(G' + j\omega C')}} = \sqrt{\frac{j\omega L'}{j\omega C'}}$$

$$Z_0 = \sqrt{\frac{L'}{C'}}$$

lossless line

$$\alpha = 0$$

$$\beta = \omega\sqrt{L'C'}$$

$\beta = 2\pi/\lambda$ ⇒ $\lambda = 2\pi/\beta = \dfrac{1}{\omega\sqrt{L'C'}}$

$$V_p = \omega/\beta = \frac{1}{\sqrt{L'C'}}$$

- For TEM transmission line :

$L'C' = \mu\varepsilon$

$$V_p = \frac{1}{\sqrt{L'C'}} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}}$$

$$\beta = \omega\sqrt{L'C'} = \omega\sqrt{\mu\varepsilon}$$

- summary :

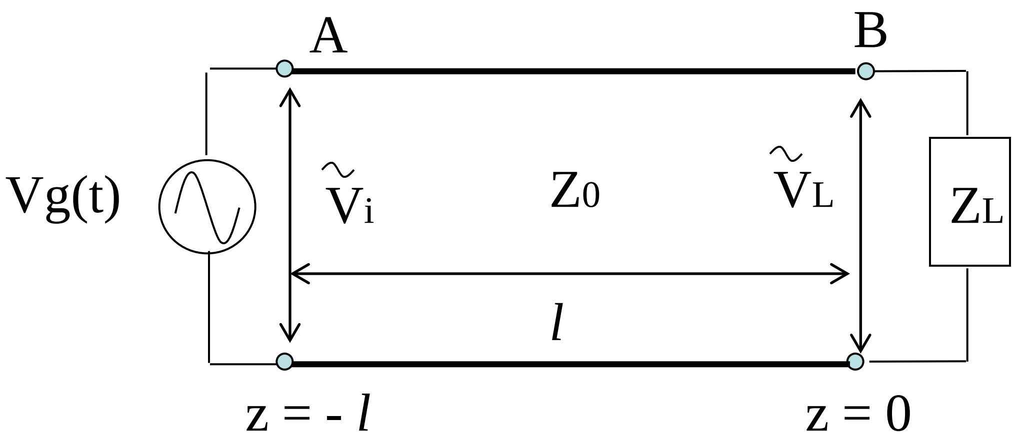
$$\begin{cases} \tilde{V}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \\ \tilde{i}(z) = I_0^+ e^{-j\beta z} + I_0^- e^{j\beta z} \end{cases}$$

$$Z_0 = \sqrt{\frac{L'}{C'}}$$

$$V_p = \frac{1}{\sqrt{L'C'}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}}$$

$$\beta = \omega\sqrt{L'C'} = \omega\sqrt{\mu\varepsilon}$$

- Voltage reflection coefficient :



$$\begin{cases} \tilde{V}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \\ \tilde{i}(z) = \dfrac{V_0^+}{Z_0} e^{-j\beta z} - \dfrac{V_0^-}{Z_0} e^{j\beta z} \end{cases}$$

$$\tilde{V}_L = \tilde{V}(z)\Big|_{z=0} = V_0^+ + V_0^-$$

$$\tilde{i}_L = \tilde{i}(z)\Big|_{z=0} = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}$$

$$Z_L = \frac{\tilde{V}_L}{\tilde{i}_L} = \frac{V_0^+ + V_0^-}{\dfrac{V_0^+}{Z_0} - \dfrac{V_0^-}{Z_0}} \quad \Rightarrow \quad \frac{V_0^+}{V_0^-} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

- Voltage reflection coefficient :

$$\Gamma \equiv \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

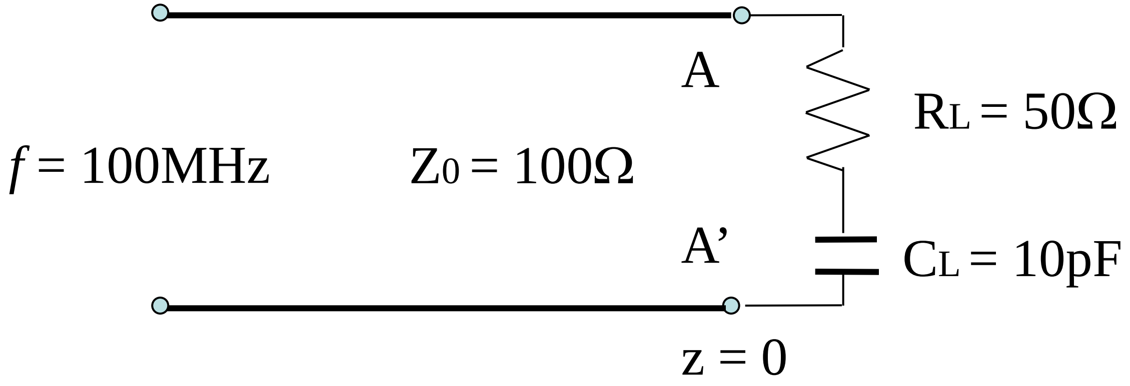
- Current reflection coefficient :

$$\Gamma_i \equiv \frac{i_0^-}{i_0^+} = -\frac{V_0^-}{V_0^+} = -\Gamma$$

- Notes :

1. $|\Gamma| \leq 1$, how to prove it?
2. If $Z_L = Z_0$, $\Gamma = 0$. Impedance match, no reflection from the load Z_L.

- <u>An example :</u>



$Z_L = R_L + j/\omega C_L = 50 - j159$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

16.360 Lecture 6

- Standing wave
- Input impedance

$$\begin{cases} \tilde{V}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \\ \tilde{i}(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{j\beta z} \end{cases}$$

with $\Gamma = \frac{V_0^-}{V_0^+}$

$$\begin{cases} \tilde{V}(z) = V_0^+ (e^{-j\beta z} + \Gamma e^{j\beta z}) \\ \tilde{i}(z) = \frac{V_0^+}{Z_0} (e^{-j\beta z} - \Gamma e^{j\beta z}) \end{cases}$$

$$|\tilde{V}(z)| = |V_0^+| \, | e^{-j\beta z} + |\Gamma| e^{j\theta_r} e^{j\beta z} |$$

$$= |V_0^+| \, [1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta z + \theta_r)]^{1/2}$$

- Standing wave

$$\begin{cases} \tilde{V}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \\ \tilde{i}(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{j\beta z} \end{cases}$$

with $\Gamma = \frac{V_0^-}{V_0^+}$

$$\begin{cases} \tilde{V}(z) = V_0^+ (e^{-j\beta z} + \Gamma e^{j\beta z}) \\ \tilde{i}(z) = \frac{V_0^+}{Z_0} (e^{-j\beta z} - \Gamma e^{j\beta z}) \end{cases}$$

$$|\tilde{V}(z)| = |V_0^+| \, | e^{-j\beta z} + |\Gamma| e^{j\theta_r} e^{j\beta z}|$$

$$= |V_0^+| \, [1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta z + \theta_r)]^{1/2}$$

- Standing wave

$$\begin{cases} \tilde{V}(z) = V_0^+(e^{-j\beta z} + \Gamma e^{j\beta z}) \\ \tilde{i}(z) = \dfrac{V_0^+}{Z_0}(e^{-j\beta z} - \Gamma e^{j\beta z}) \end{cases}$$

$$|\tilde{i}(z)| = |V_0^+|/|Z0||\, e^{-j\beta z} - |\Gamma| e^{j\theta_r} e^{j\beta z}|$$

$$= |V_0^+|/|Z_0|\,[1 + |\Gamma|^2 - 2|\Gamma|\cos(2\beta z + \theta_r)]^{1/2}$$

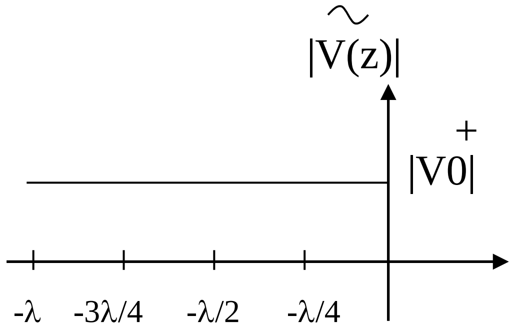
$$|\tilde{V}(z)| = |V_0^+|\,[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta z + \theta_r)]^{1/2}$$

$$|\tilde{V}(z)| = |V_0^+| \left[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta z + \theta_r)\right]^{1/2}$$

<u>Special cases</u>

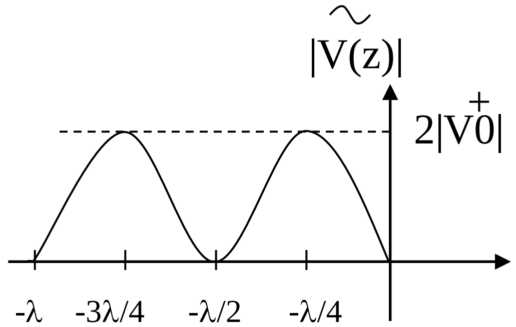
1. $Z_L = Z_0, \Gamma = 0$

$$|\tilde{V}(z)| = |V_0^+|$$



2. $Z_L = 0$, short circuit, $\Gamma = -1$

$$|\tilde{V}(z)| = |V_0^+| \left[2 + 2\cos(2\beta z + \pi)\right]^{1/2}$$

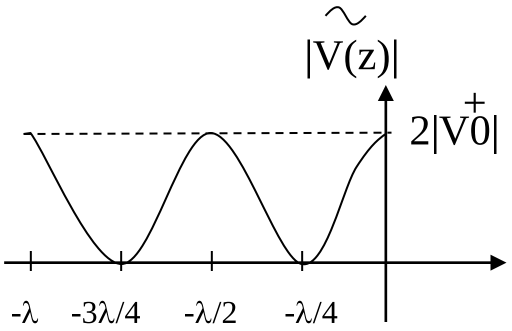


$$|\tilde{V}(z)| = |V_0^+| \left[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta z + \theta_r)\right]^{1/2}$$

Special cases

3. $Z_L = \infty$, open circuit, $\Gamma = 1$

$$|\tilde{V}(z)| = |V_0^+| \left[2 + 2\cos(2\beta z)\right]^{1/2}$$



- Voltage maximum

$$|\tilde{V}(z)| = |V_0^+| \left[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta z + \theta_r)\right]^{1/2}$$

$$|\tilde{V}(z)|_{max} = |V_0^+| \left[1 + |\Gamma|\right],$$ when $2\beta z + \theta_r = 2n\pi$.

$$-z = \lambda\theta_r/4\pi + n\lambda/2$$

$n = 1, 2, 3, \ldots,$ if $\theta_r < 0$

$n = 0, 1, 2, 3, \ldots,$ if $\theta_r >= 0$

- Voltage minimum

$$|\tilde{V}(z)| = |V_0^+| \left[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta z + \theta_r)\right]^{1/2}$$

$$|\tilde{V}(z)|_{min} = |V_0^+| \left[1 - |\Gamma|\right], \quad \text{when } 2\beta z + \theta_r = (2n+1)\pi.$$

$$-z = \lambda\theta_r/4\pi + n\lambda/2 + \lambda/4$$

Note:

voltage minimums occur $\lambda/4$ away from voltage maximum, because of the $2\beta z$, the special frequency doubled.

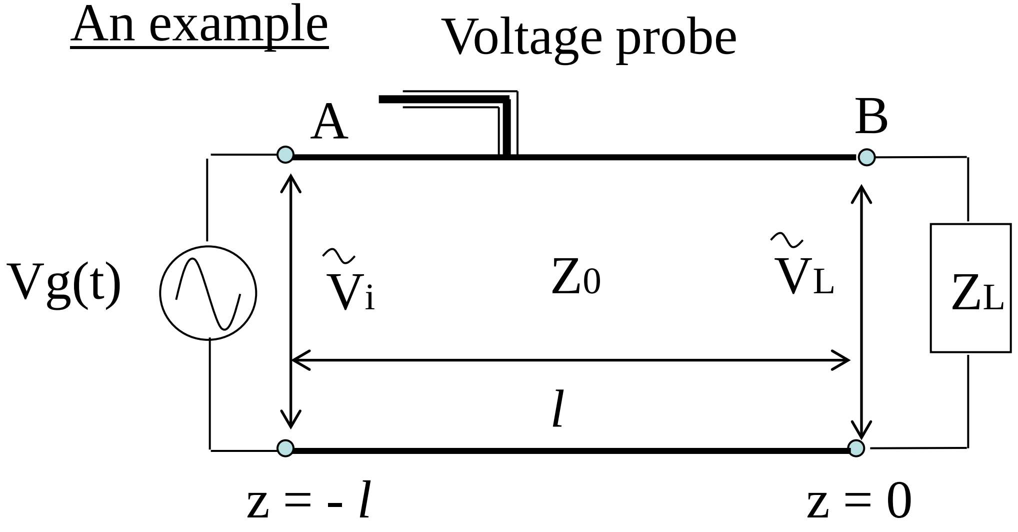
- <u>Voltage standing-wave ratio VSWR or SWR</u>

$$S \equiv \frac{|\tilde{V}(z)|_{max}}{|\tilde{V}(z)|_{min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

$S = 1$, when $\Gamma = 0$,

$S = \infty$, when $|\Gamma| = 1$,

- An example



$S = 3$, $Z_0 = 50\Omega$, $\Delta l_{min} = 30cm$, $l_{min} = 12cm$, $Z_L = ?$

Solution:

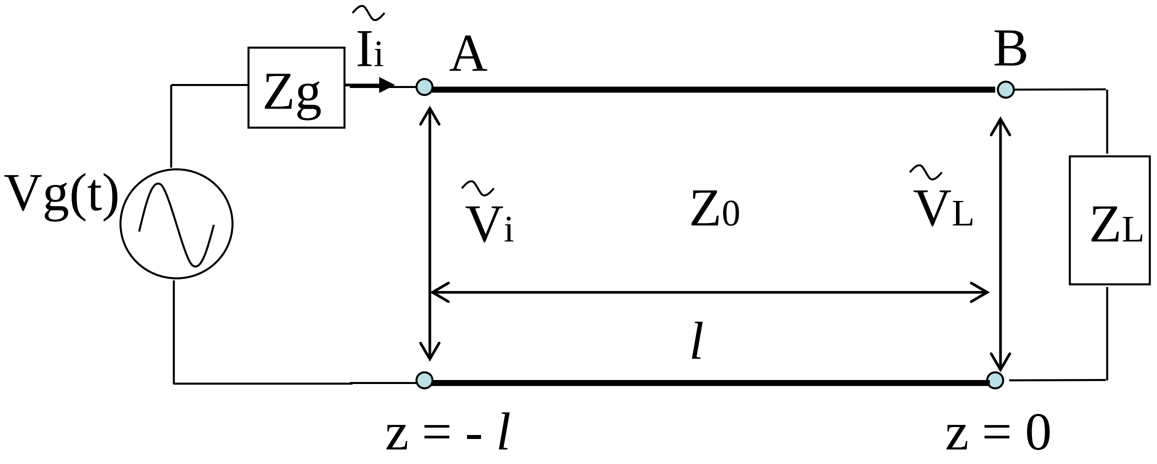
$\Delta l_{min} = 30cm, \Rightarrow \lambda = 0.6\text{m},$

$S = 3, \Rightarrow |\Gamma| = 0.5,$

$-2\beta l_{min} + \theta_r = -\pi, \Rightarrow \theta_r = -36^o,$

$\Rightarrow \Gamma$, and Z_L.

- Input impudence



$$Z_{in}(z) = \frac{\tilde{V}(z)}{\tilde{I}(z)}$$

$$= \frac{V_0^+(e^{-j\beta z} + \Gamma e^{j\beta z})}{V_0^+(e^{-j\beta z} - \Gamma e^{j\beta z})} Z_0 = \frac{(1 + \Gamma e^{j2\beta z})}{(1 - \Gamma e^{j2\beta z})} Z_0$$

$$Z_{in}(-l) = \frac{(1 + \Gamma e^{-j2\beta l})}{(1 - \Gamma e^{-j2\beta l})} Z_0$$

An example

A 1.05-GHz generator circuit with series impedance Zg = 10-Ω and voltage source given by Vg(t) = 10 sin(ωt +30º) is connected to a load ZL = 100 +j5-Ω through a 50-Ω, 67-cm long lossless transmission line. The phase velocity is 0.7c. Find V(z,t) and *i*(z,t) on the line.

Solution:

Since, $V_p = f\lambda$, $\lambda = V_p/f = 0.7c/1.05\text{GHz} = 0.2\text{m}$.

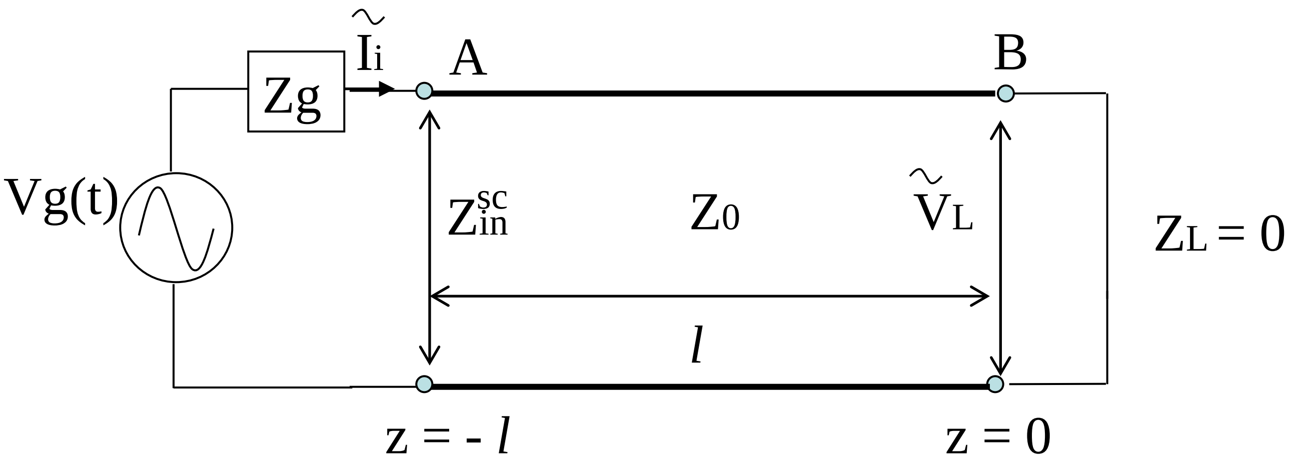
$\beta = 2\pi/\lambda$, $\beta = 10\pi$.

$\Gamma = (Z_L - Z_0)/(Z_L + Z_0)$, $\Gamma = 0.45\exp(j26.6^\circ)$

$$Z_{in}(-l) = \frac{(1+\Gamma e^{-j2\beta l})}{(1-\Gamma e^{-j2\beta l})} Z_0 = 21.9 + j17.4\ \Omega$$

$$V_0^+[\exp(-j\beta l) + \Gamma\exp(j\beta l)] = \frac{Z_{in}(-l)}{Z_{in}(-l) + Z_g}\tilde{V}_g$$

short circuit line



$Z_L = 0$, $\Gamma = -1$, $S = \infty$

$$\begin{cases} \tilde{V}(z) = V_0^+(e^{-j\beta z} - e^{j\beta z}) = -2jV_0^+\sin(\beta z) \\ \tilde{i}(z) = \dfrac{V_0^+}{Z_0}(e^{-j\beta z} + e^{j\beta z}) = 2V_0^+\cos(\beta z)/Z_0 \end{cases}$$

$$Z_{in} = \frac{\tilde{V}(-l)}{\tilde{i}(-l)} = jZ_0\tan(\beta l)$$

short circuit line

$$Z_{in} = \frac{\tilde{V}(-l)}{\tilde{i}(-l)} = jZ_0\tan(\beta l)$$

- If $\tan(\beta l) >= 0$, the line appears inductive, $j\omega L_{eq} = jZ_0\tan(\beta l)$,

- If $\tan(\beta l) <= 0$, the line appears capacitive, $1/j\omega C_{eq} = jZ_0\tan(\beta l)$,

- The minimum length results in transmission line as a capacitor:

$$l = 1/\beta[\pi - \tan^{-1}(1/\omega C_{eq}Z_0)],$$

An example:

Choose the length of a shorted 50-Ω lossless line such that its input impedance at 2.25 GHz is equivalent to the reactance of a capacitor with capacitance Ceq = 4pF. The wave phase velocity on the line is 0.75c.

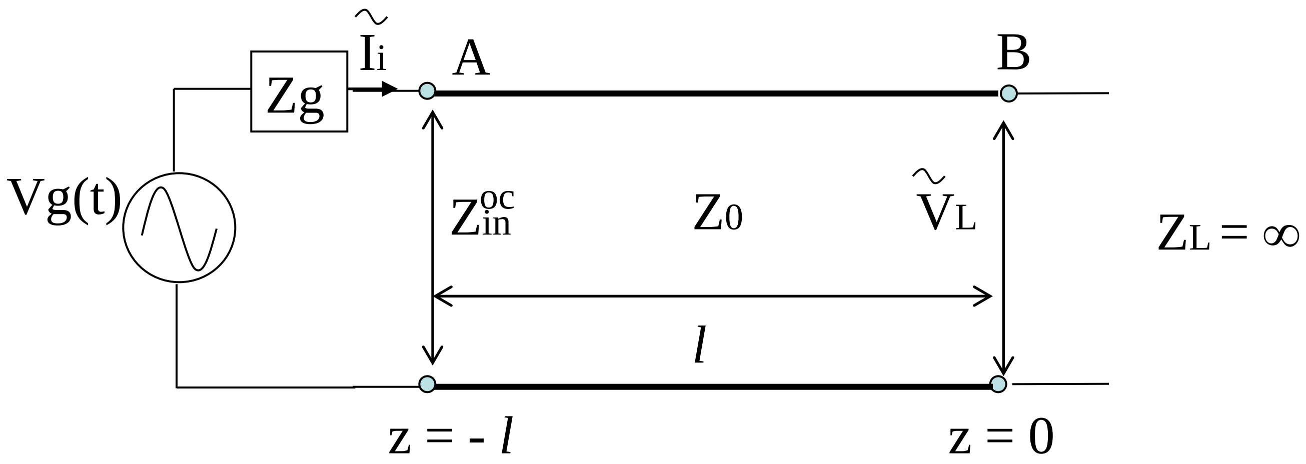
Solution:

$$V_p = \lambda f, \Rightarrow \beta = 2\pi/\lambda = 2\pi f/V_p = 62.8 \text{ (rad/m)}$$

$$\tan(\beta l) = -1/\omega C_{eq} Z_0 = -0.354,$$

$$\beta l = \tan^{-1}(-0.354) + n\pi,$$
$$= -0.34 + n\pi,$$

open circuit line



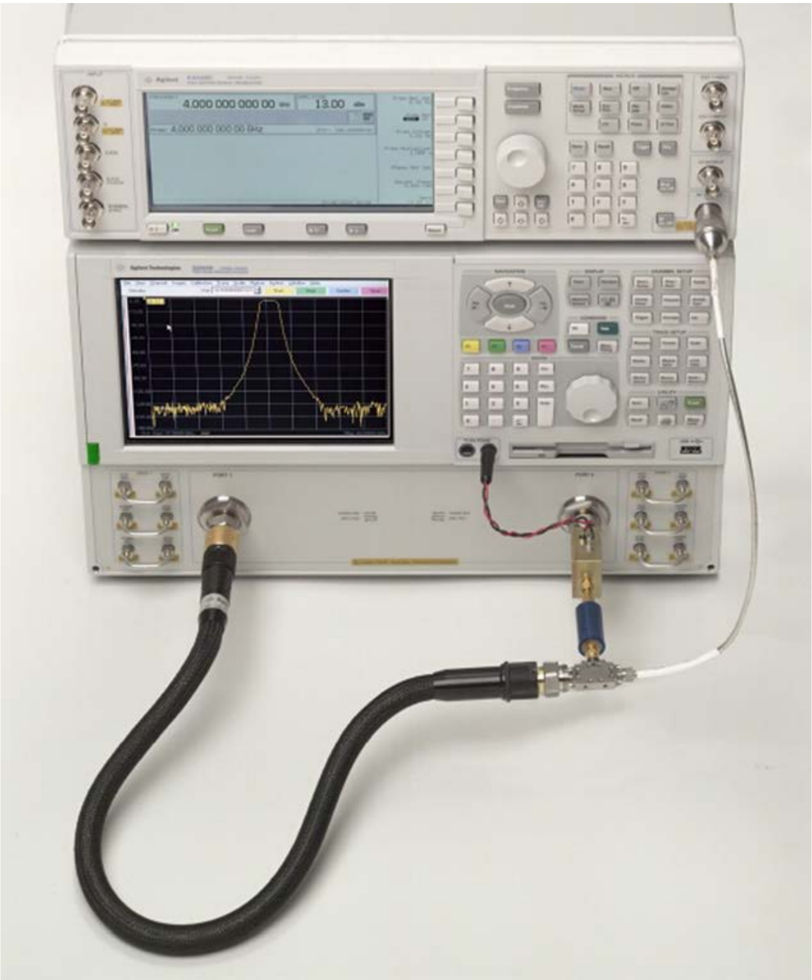
$Z_L = 0, \Gamma = 1, S = \infty$

$$\begin{cases} \tilde{V}(z) = V_0(e^{-j\beta z} + e^{j\beta z}) = 2V_0^+\cos(\beta z) \\ \tilde{i}(z) = \dfrac{V_0^+}{Z_0}(e^{-j\beta z} - e^{j\beta z}) = 2jV_0^+\sin(\beta z)/Z_0 \end{cases}$$

$$Z_{in}^{oc} = \frac{\tilde{V}(-l)}{\tilde{i}(-l)} = -jZ_0\cot(\beta l)$$

Application for short-circuit and open-circuit

- Network analyzer



- Measure S paremeters
- Measure Z_{in}^{sc} and Z_{in}^{oc}
- Calculate Z_0

$$Z_{in}^{sc} = jZ_0\tan(\beta l)$$
$$Z_{in}^{oc} = -jZ_0\cot(\beta l)$$

$$Z_0 = \sqrt{Z_{in}^{sc} Z_{in}^{oc}}$$

- Calculate βl

$$Z_0 = -j\sqrt{\frac{Z_{in}^{sc}}{Z_{in}^{oc}}}$$

Line of length $l = n\lambda/2$

$\tan(\beta l) = \tan((2\pi/\lambda)(n\lambda/2)) = 0,$

$$Z_{in}(-l) = \frac{(1+\Gamma e^{-j2\beta l})}{(1-\Gamma e^{-j2\beta l})} Z_0 = Z_L$$

Any multiple of half-wavelength line doesn't modify the load impedance.

Quarter-wave transformer $l = \lambda/4 + n\lambda/2$

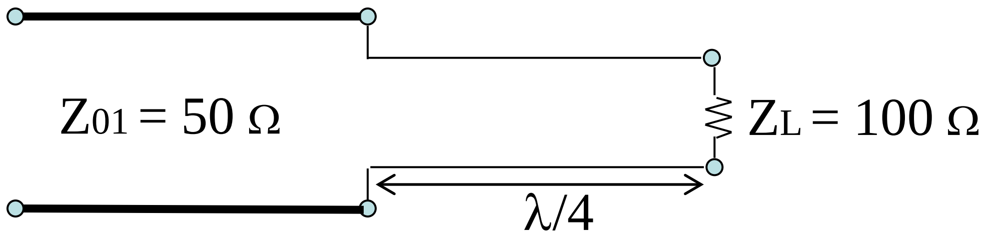
$$\beta l = (2\pi/\lambda)(\lambda/4 + n\lambda/2) = \pi/2 ,$$

$$Z_{in}(-l) = \frac{(1+\Gamma e^{-j2\beta l})}{(1-\Gamma e^{-j2\beta l})} Z_0 = \frac{(1+\Gamma e^{-j\pi})}{(1-\Gamma e^{-j\pi})} Z_0 = \frac{(1-\Gamma)}{(1+\Gamma)} Z_0$$

$$= Z_0^2/Z_L$$

An example:

A 50-Ω lossless tarnsmission is to be matched to a resistive load impedance with Z_L = 100 Ω via a quarter-wave section, thereby eliminating reflections along the feed line. Find the characteristic impedance of the quarter-wave tarnsformer.



$$Z_{in} = Z_0^2/Z_L = 50\ \Omega$$

$$Z_0 = (Z_{in}Z_L)^{1/2} = (50*100)^{1/2}$$

Matched transmission line:

1. $Z_L = Z_0$
2. $\Gamma = 0$
3. All incident power is delivered to the load.

- Instantaneous power
- Time-average power

$$\begin{cases} \tilde{V}(z) = V_0^+(e^{-j\beta z} + \Gamma e^{j\beta z}) \\ \tilde{i}(z) = \dfrac{V_0^+}{Z_0}(e^{-j\beta z} - \Gamma e^{j\beta z}) \end{cases}$$

At load z = 0, the incident and reflected voltages and currents:

$$\tilde{V}^i = V_0^+ \qquad \tilde{i}^i = \frac{V_0^+}{Z_0}$$

$$\tilde{V}^r = V_0^- \qquad \tilde{i}^r = \frac{V_0^-}{Z_0}$$

- Instantaneous power

$$P^i(t) = v(t)\, i(t) = \mathrm{Re}[\tilde{V}^i \exp(j\omega t)]\, \mathrm{Re}[\tilde{i}^i \exp(j\omega t)]$$

$$= \mathrm{Re}[|V_0^+| \exp(j\phi^+)\exp(j\omega t)]\, \mathrm{Re}[|V_0^+|/Z_0 \exp(j\phi^+)\exp(j\omega t)]$$

$$= (|V_0^+|^2/Z_0) \cos^2(\omega t + \phi^+)$$

$$P^r(t) = v(t)\, i(t) = \mathrm{Re}[\tilde{V}^r \exp(j\omega t)]\, \mathrm{Re}[\tilde{i}^r \exp(j\omega t)]$$

$$= \mathrm{Re}[|V_0^-| \exp(j\phi^+)\exp(j\omega t)]\, \mathrm{Re}[|V_0^-|/Z_0 \exp(j\phi^+)\exp(j\omega t)]$$

$$= -\,|\Gamma|^2(|V_0^+|^2/Z_0) \cos^2(\omega t + \phi^+ + \phi_r)$$

- Time-average

<u>Time-domain approach:</u>

$$P^{i}_{av} = \frac{1}{T}\int_0^T P^{i}(t)dt = \frac{\omega}{2\pi}\int_0^T (|V_0^+|^2/Z_0)\cos^2(\omega t + \phi^+)dt$$

$$= (|V_0^+|^2/2Z_0)$$

$$P^{r}_{av} = -|\Gamma|^2 (|V_0^+|^2/2Z_0)$$

<u>Net average power:</u>

$$P_{av} = P^{i}_{av} + P^{r}_{av}$$

$$= (1-|\Gamma|^2)(|V_0^+|^2/2Z_0)$$

- Time-average

Phasor-domain approach

$$P_{av} = (1/2)\text{Re}[\tilde{V}\,\tilde{i}^*]$$

$$P^i_{av} = (1/2)\,\text{Re}[V_0^+ V_0^{+*}/Z_0] = (|V_0^+|^2/2Z_0)$$

$$P^r_{av} = -|\Gamma|^2\,(|V_0^+|^2/2Z_0)$$

$$P_{av} = (1-|\Gamma|^2)\,(|V_0^+|^2/2Z_0)$$

Normalized admittance

$$y = \frac{Y}{Y_0} = \frac{G}{Y_0} + j\frac{B}{Y_0} = g + jb,$$

$$g = GZ_0, \quad b = BZ_0, \quad y = \frac{Y}{Y_0} = \frac{Z_0}{Z} = \frac{1}{z},$$

$$y_L = \frac{1}{Z_L} = \frac{1-\Gamma}{1+\Gamma}, \quad y_{in} = \frac{1}{Z_{in}} = \frac{1-\Gamma e^{-j\beta l}}{1+\Gamma e^{-j\beta l}},$$

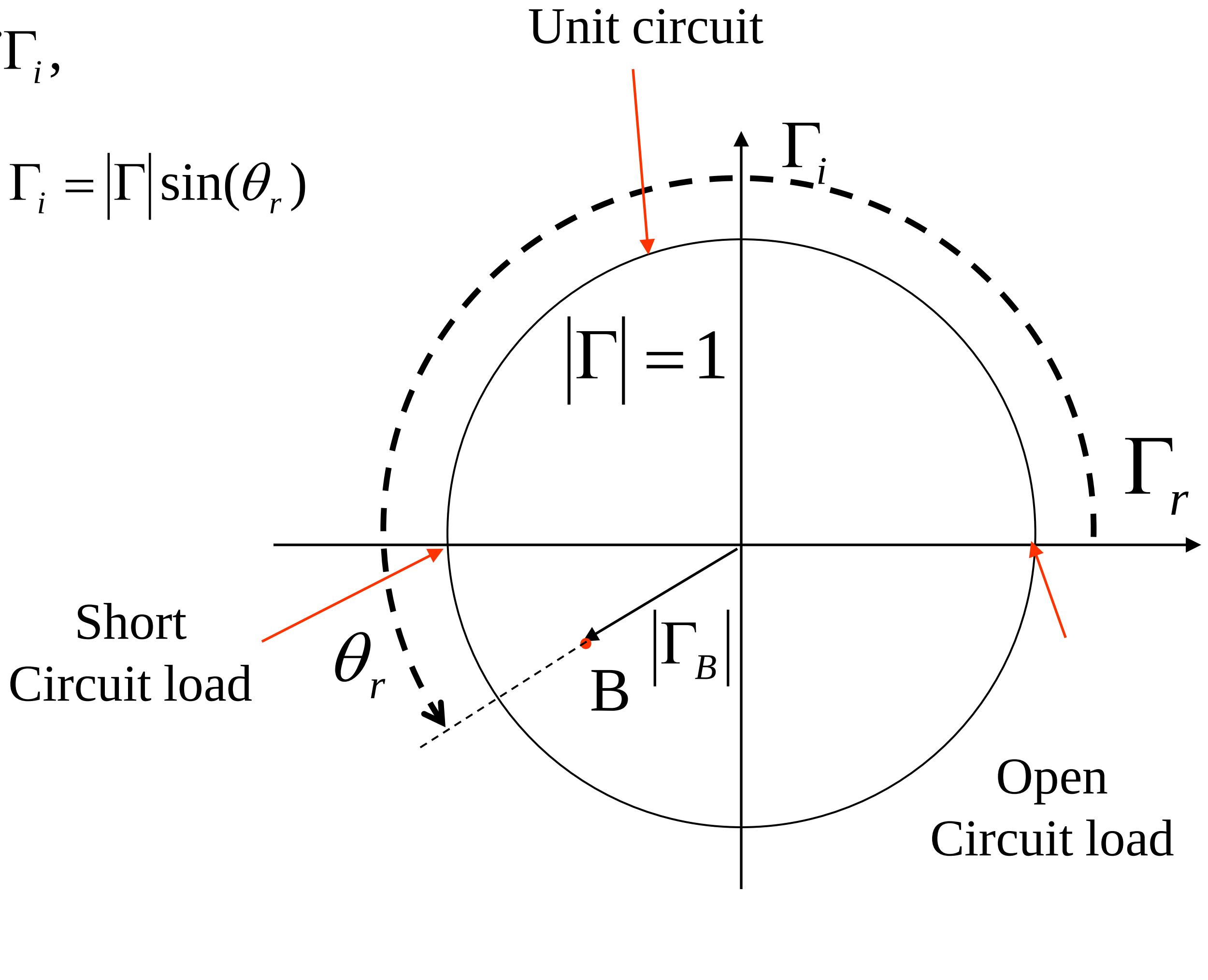
$$z_{in}(l = \lambda / 4) = \frac{1-\Gamma}{1+\Gamma} = y_L,$$

z and y are directly opposite each other on Smith Chart

Parameter equations

$$\Gamma = |\Gamma| e^{j\theta_r} = \Gamma_r + j\Gamma_i,$$

$$\Gamma_r = |\Gamma|\cos(\theta_r) \qquad \Gamma_i = |\Gamma|\sin(\theta_r)$$



Normalized impedance

$$z_L = \frac{Z_L}{Z_0}, \qquad \Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_L / Z_0 - 1}{Z_L / Z_0 + 1} = \frac{z_L - 1}{z_L + 1},$$

$$z_L = \frac{1+\Gamma}{1-\Gamma}, \qquad z_L = r_L + jx_L, \; r_L + jx_L = \frac{(1+\Gamma_r) + j\Gamma_i}{(1-\Gamma_r) - j\Gamma_i}$$

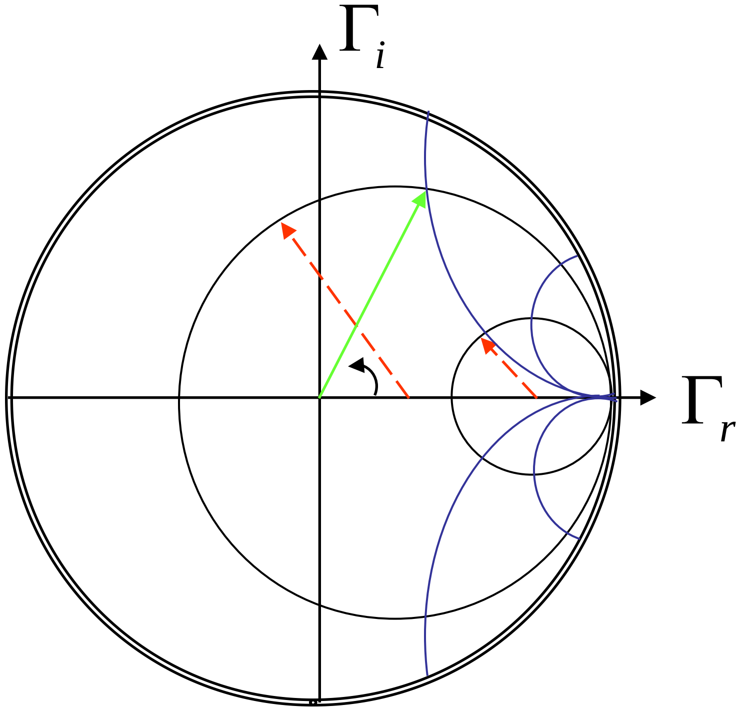
Parameter equations

$$r_L = \frac{1-\Gamma_r^{\,2} - \Gamma_i^{\,2}}{(1-\Gamma_r)^2 + \Gamma_i^{\,2}} \qquad x_L = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^{\,2}},$$

$$(\Gamma_r - \frac{r_L}{1+r_L})^2 + \Gamma_i^{\,2} = 1, \quad (\Gamma_r - 1)^2 + (\Gamma_i - \frac{1}{x_L})^2 = (\frac{1}{x_L})^2,$$

Parameter equations

$$(\Gamma_r - \frac{r_L}{1+r_L})^2 + \Gamma_i^{\,2} = 1, \quad (\Gamma_r - 1)^2 + (\Gamma_i - \frac{1}{x_L})^2 = (\frac{1}{x_L})^2,$$



An example

$$z_L = 0.5 - j2,$$

Smith Chart

Input impedance

$$z_{in} = \frac{1+\Gamma e^{-j\beta l}}{1-\Gamma e^{-j\beta l}}, \qquad \Gamma_l = \Gamma e^{-j\beta l} = |\Gamma| e^{j(\theta_r - \beta l)},$$

Wavelength toward generator (WTG)

Smith Chart

An example

$Z_L = 100 - j50,$ find Zin (-0.1λ)

$z_L = 2 - j,$

Smith Chart

$$S = \frac{1+|\Gamma|}{1-|\Gamma|},$$ Constant |Γ| circle, SWR Circle

SWR, voltage maximum and minimum

$z_L = r_L + jx_L$, If $x_L = 0$,

Smith Chart

$$S = \frac{1+|\Gamma|}{1-|\Gamma|}, \quad \Gamma_L = \frac{z_L - 1}{z_L + 1} = \frac{r_L - 1}{r_L + 1}, \quad r_L = \frac{1+\Gamma_r}{1-\Gamma_r}, \quad S = r_L,$$

Recall:

$$|\tilde{V}(z)|_{max} = |V_0^+|\,[1 + |\Gamma|], \text{ when } 2\beta z + \theta_r = 2n\pi.$$

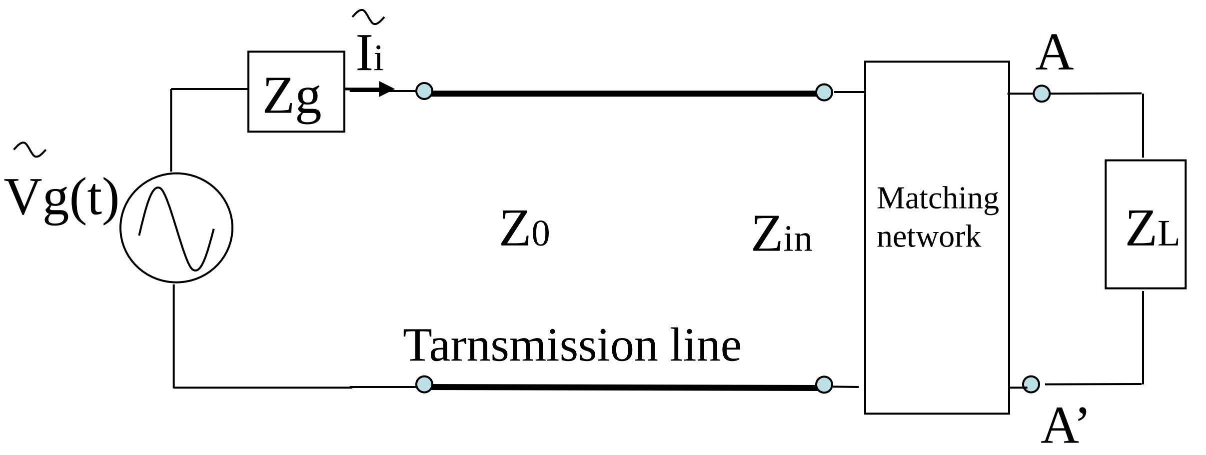
$$|\tilde{V}(z)|_{min} = |V_0^+|\,[1 - |\Gamma|], \text{ when } 2\beta z + \theta_r = (2n+1)\pi.$$

An example

A 50-Ω lossless line is terminated in a load ZL = (25+j50)Ω. Use the smith chart to find a) voltage reflection coefficient, b) the voltage standing-wave ratio, c) the distances of the first voltage maximum and first voltage minimum from the load, d) the input impedance of the line, given the line is 3.3λ, and e) the input admittance of the line.

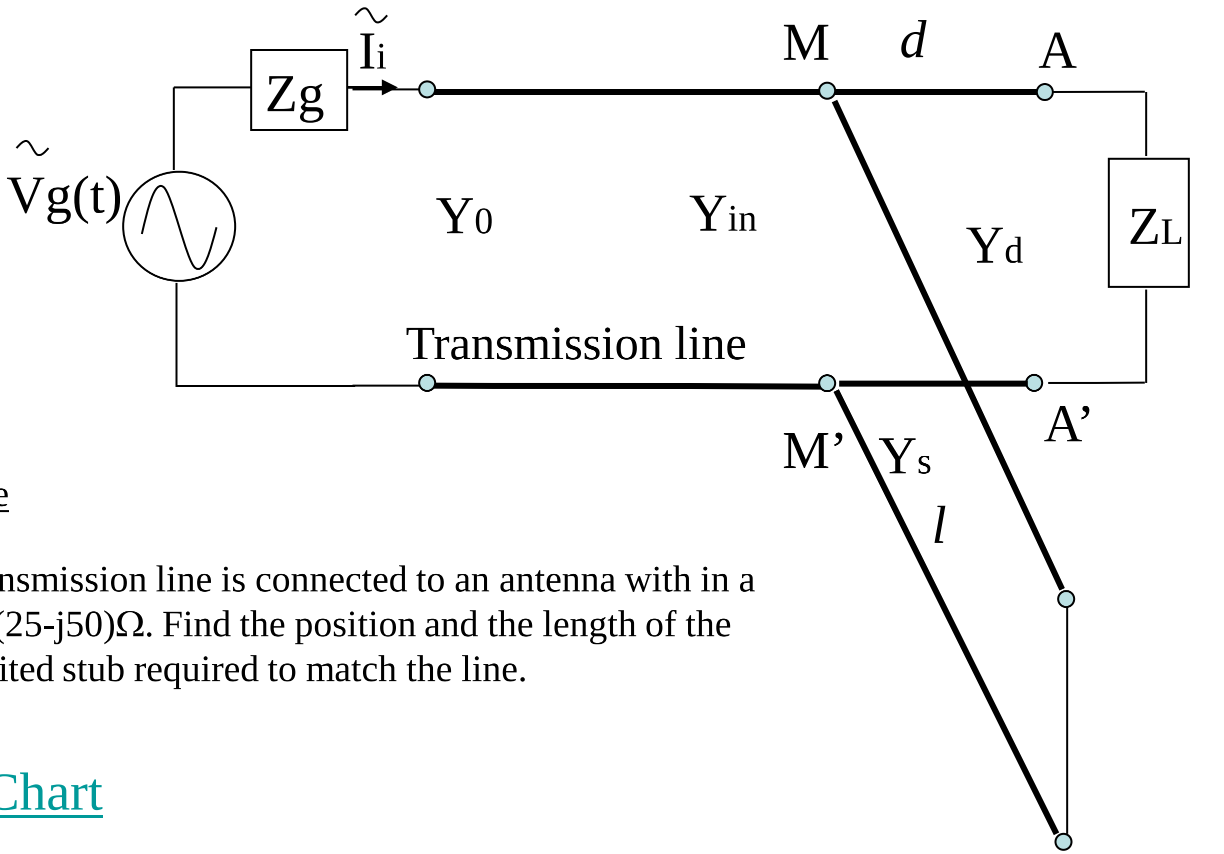
Smith Chart

• impedance matching



$Z_{in} = Z_0$

• single-stub impedance matching network

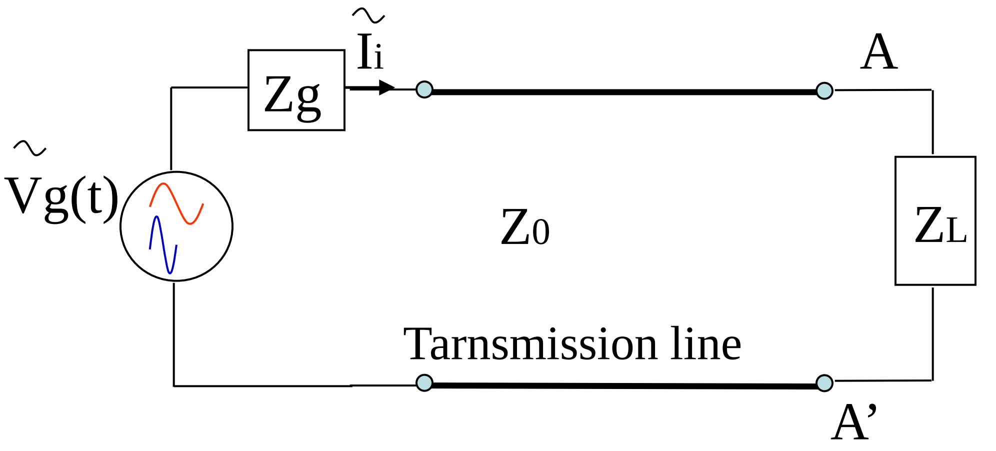


An example

A 50-Ω transmission line is connected to an antenna with in a load ZL = (25-j50)Ω. Find the position and the length of the short-circuited stub required to match the line.

Smith Chart

- Transient on transmission line



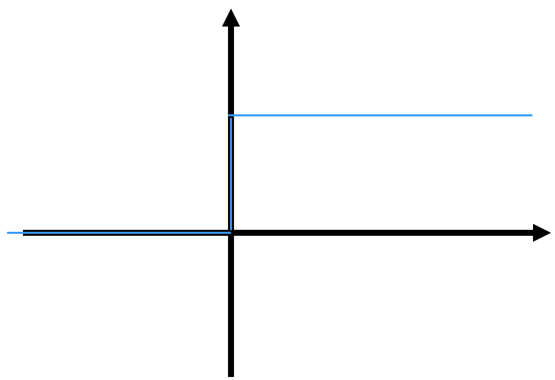
If ω1, ω2,…, ωn are transmitted on the transmission line at the same time, each frequency has its own location of voltage distribution. The total voltage V(z) is the sum of all these $V_{\omega i}(z)$.

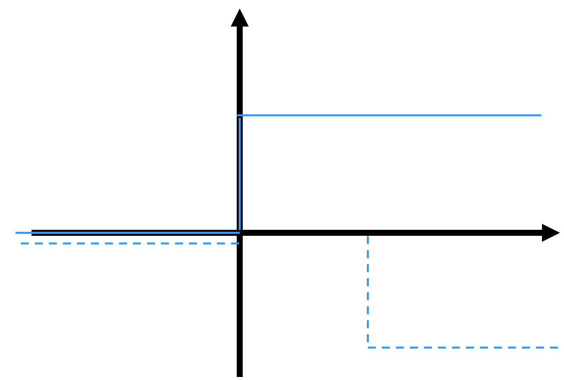
$$V(z) = \sum_{i=\omega_1}^{\omega_n} V_{\omega_i}(z)$$

Step function and pulse function

• step function U(t)

U(t) = 1, if t>=0; U(t) = 0, if t<0

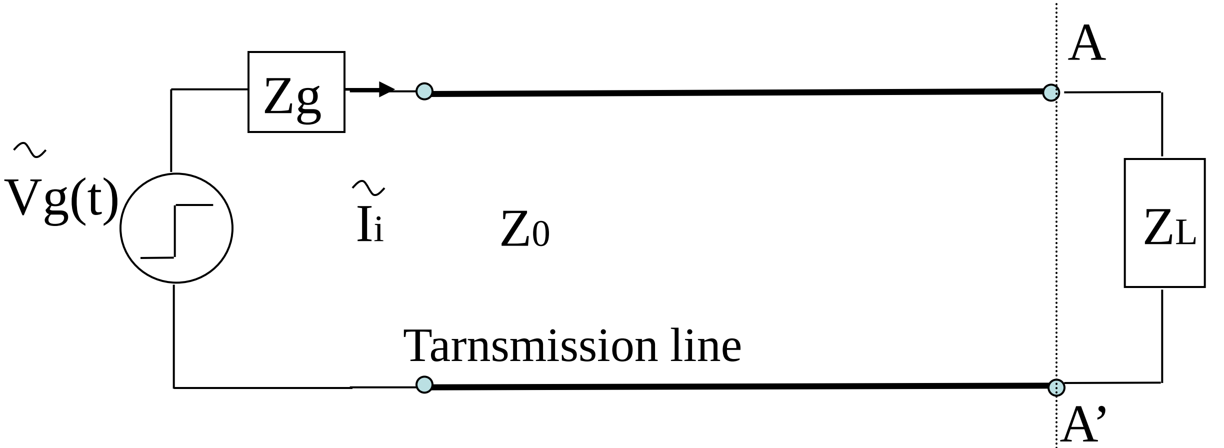




• single pulse function V(t)

V(t) = U(t) – U(t-t0),

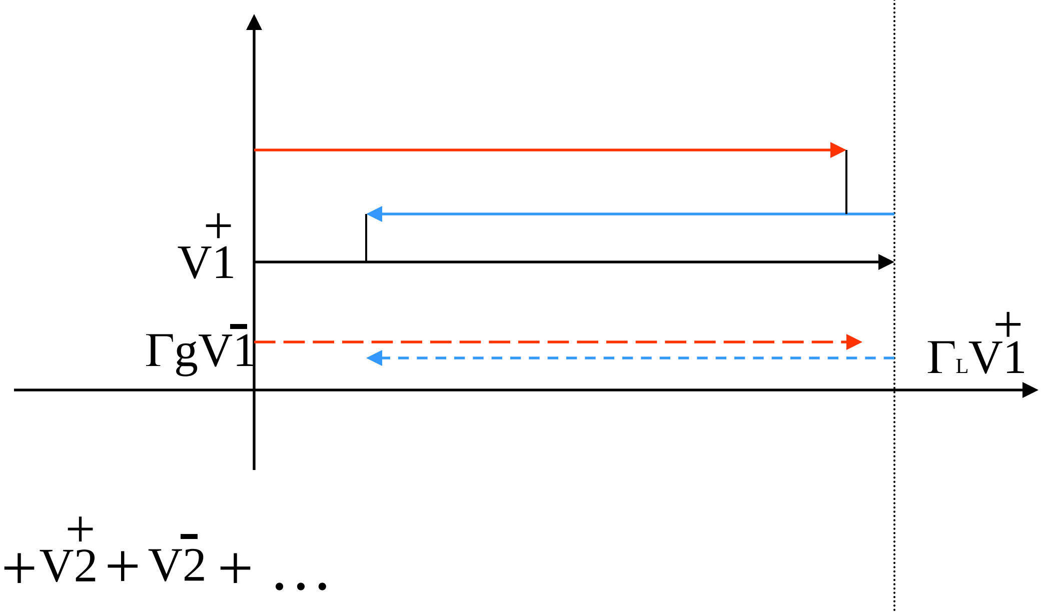
- transient of a step function



$V_1^- = \Gamma_L V_1^+$

$V_2^+ = \Gamma_g V_1^-$

$V_2^- = \Gamma_L V_2^+$

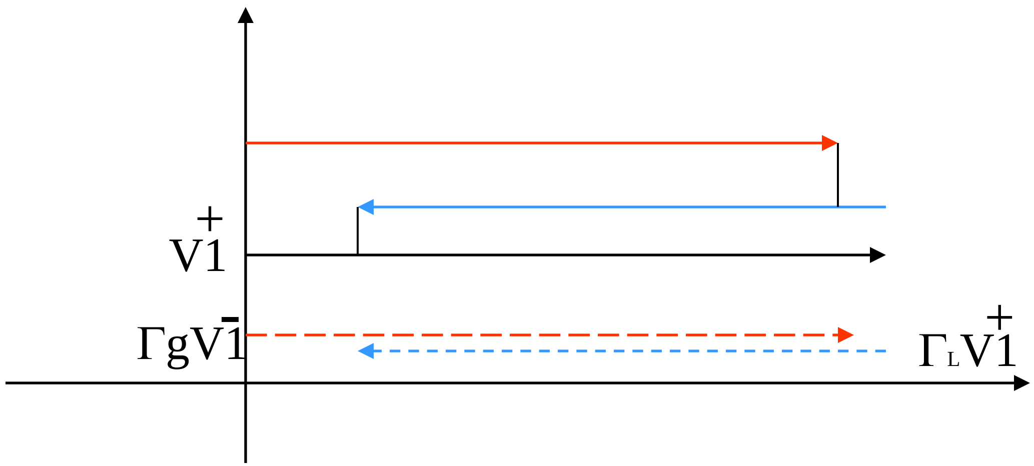


$$V = V_1^+ + V_1^- + V_2^+ + V_2^- + \ldots$$

$V_1^- = \Gamma_L V_1^+$

$V_2^+ = \Gamma_g V_1^-$

$V_2^- = \Gamma_L V_2^+$



$$V = V_1^+ + V_1^- + V_2^+ + V_2^- + \ldots$$

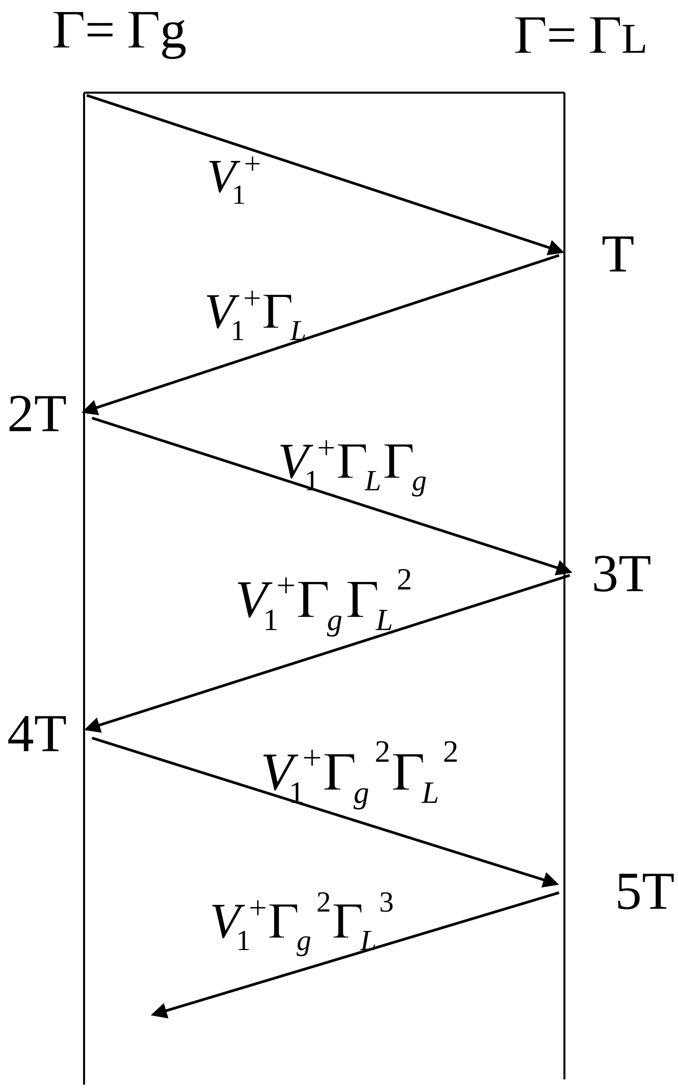
$$= V_1^+ + \Gamma_L V_1^+ + \Gamma_g(\Gamma_L V_1^+) + \Gamma_L(\Gamma_g(\Gamma_L V_1^+)) + \ldots$$

$$= V_1^+(1+\Gamma_L) + \Gamma_g \Gamma_L (1+\Gamma_L V_1^+) + \ldots$$

$$= V_1^+(1+\Gamma_L)[1+\Gamma_g\Gamma_L + (\Gamma_g\Gamma_L)^2 + \ldots]$$

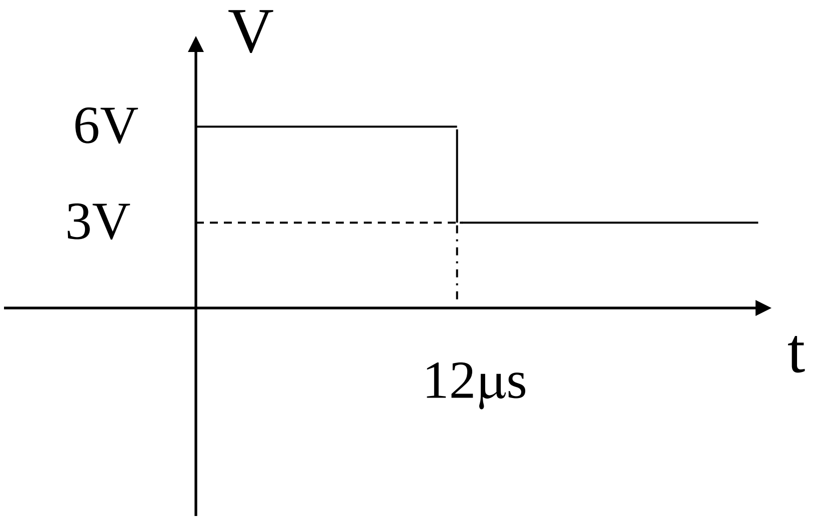
$$= V_1^+(1+\Gamma_L)[\frac{1}{1-\Gamma_g\Gamma_L}]$$

- Bounce Diagram



- An example

$Z_0 = 75\Omega$, $\varepsilon_r = 2.1$, $Vg = ?$, $Z_{lf} = ?$, $L_f = ?$



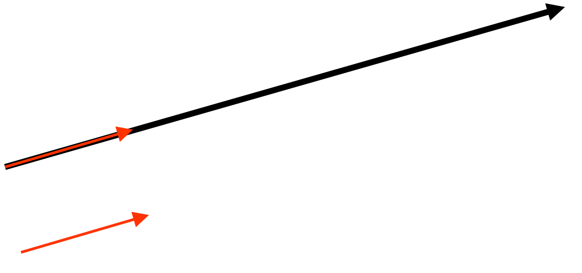
Basic Laws of Vector Algebra

Scalars:

e.g. 2 gallons, $1,000, 35ºC

Vectors:

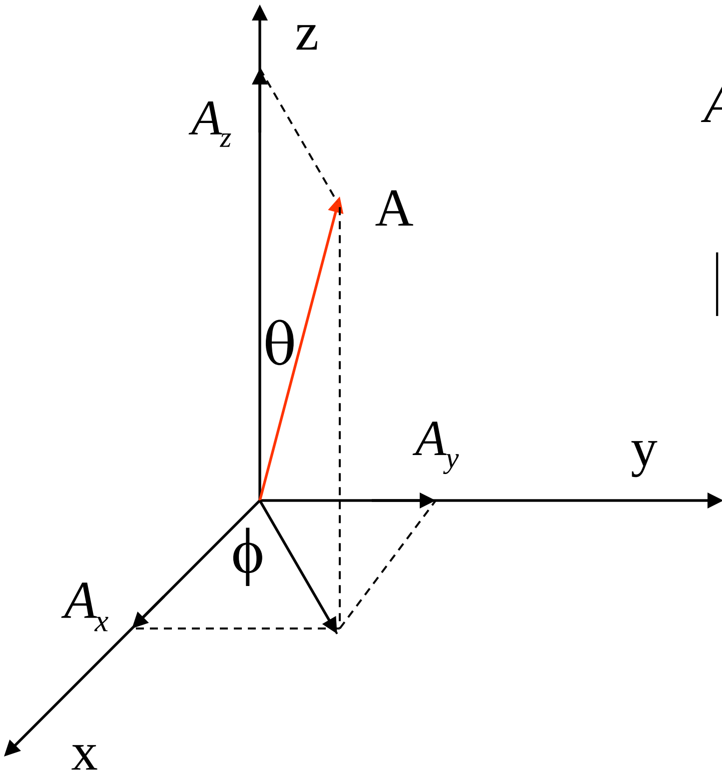
e.g. velocity: 35mph heading south
3N force toward center



$$A = \hat{a}\, A$$

$$\hat{a} = \frac{A}{|A|}$$

- Cartesian coordinate system



$$A = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

$$|A| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$A_z = |A|\cos(\theta)$$

$$A_x = |A|\sin(\theta)\cos(\phi)$$

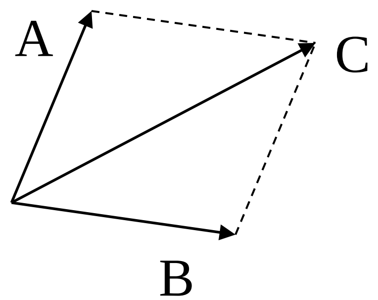
$$A_z = |A|\sin(\theta)\sin(\phi)$$

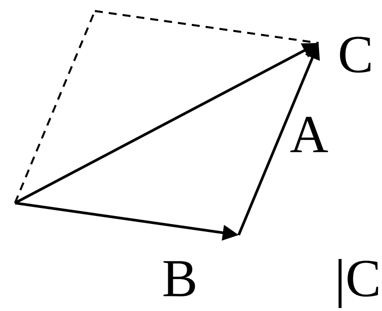
- Vector addition and subtraction

C = B+A = A +B,

$$= (\hat{x} A_x + \hat{y} A_y + \hat{z} A_z) + (\hat{x} B_x + \hat{y} B_y + \hat{z} B_z) = \hat{x}(A_x + B_x) + \hat{y}(A_y + B_x) + \hat{z}(A_z + B_z)$$

parallelogram rule



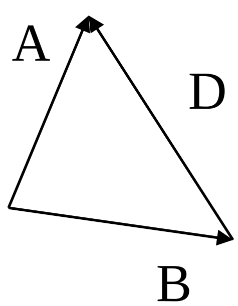


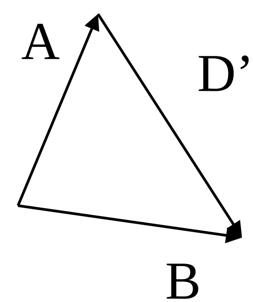
head-to-tail rule

|C| = |B+A| ≠ |A| +|B|,

D = A - B = -(B –A),

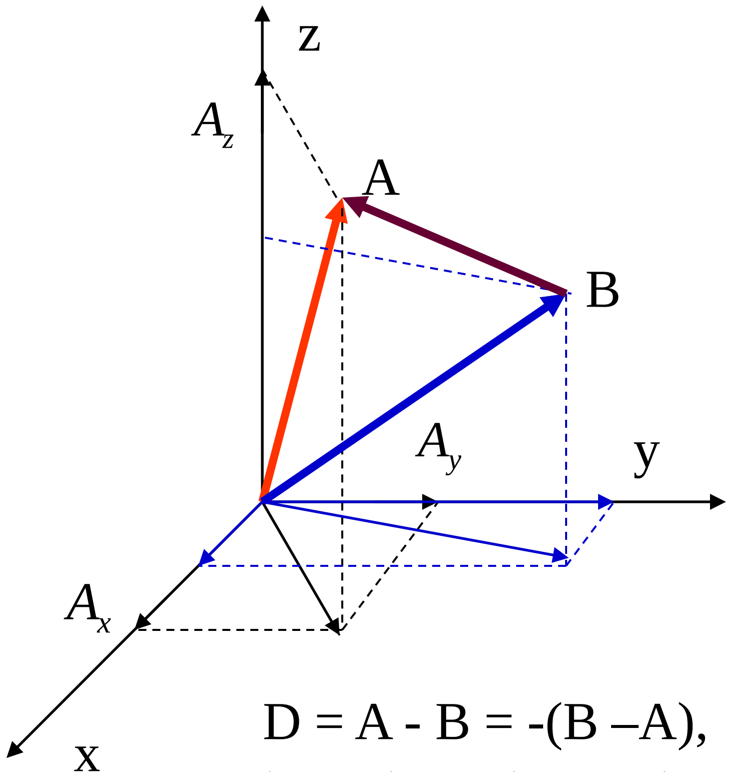
$$= (\hat{x} A_x + \hat{y} A_y + \hat{z} A_z) - (\hat{x} B_x + \hat{y} B_y + \hat{z} B_z) = \hat{x}(A_x - B_x) + \hat{y}(A_y - B_x) + \hat{z}(A_z - B_z)$$





D’ = (B-A)

- position and distance



D = A - B = -(B –A),

$$= (\hat{x} A_x + \hat{y} A_y + \hat{z} A_z) - (\hat{x} B_x + \hat{y} B_y + \hat{z} B_z) = \hat{x}(A_x - B_x) + \hat{y}(A_y - B_x) + \hat{z}(A_z - B_z)$$

$$|D| = \sqrt{(A_x - B_x)^2 + (A_y - B_x)^2 + (A_z - B_z)^2}$$

- Vector multiplication

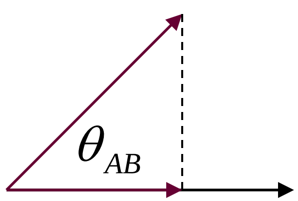
$$\vec{A} = \hat{x}(A_x) + \hat{y}(A_y) + \hat{z}(A_z)$$

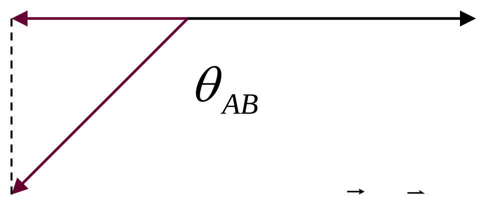
1. simple product

$$\vec{B} = k\vec{A} = \hat{x}(kA_x) + \hat{y}(kA_y) + \hat{z}(kA_z)$$

2. scalar product (dot product)

$$\vec{B} \bullet \vec{A} = |A||B|\cos\theta_{AB}$$





$$\vec{B} = \hat{x}(B_x) + \hat{y}(B_y) + \hat{z}(B_z)$$

$$\vec{A} = \hat{x}(A_x) + \hat{y}(A_y) + \hat{z}(A_z)$$

$$\vec{B} \bullet \vec{A} = [\hat{x}(B_x) + \hat{y}(B_y) + \hat{z}(B_z)] \bullet [\hat{x}(A_x) + \hat{y}(A_y) + \hat{z}(A_z)]$$

$$= [\hat{x}(B_x A_x) + \hat{y}(B_y A_y) + \hat{z}(B_z A_z)]$$

Properties of scalar product (dot product)

a) commutative property

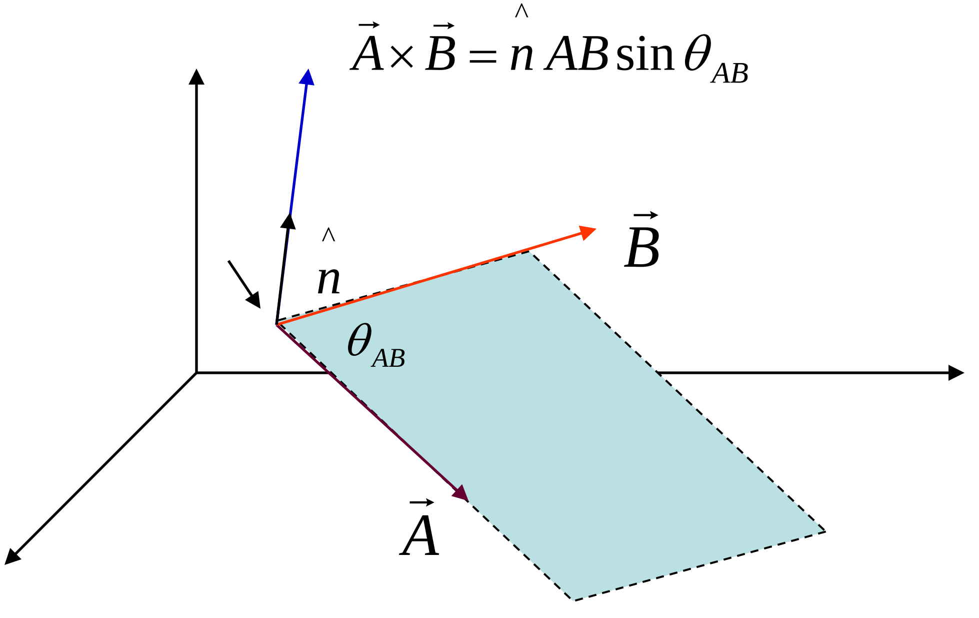
$$\vec{B}\bullet\vec{A}=\vec{A}\bullet\vec{B} \qquad \vec{A}\bullet\vec{A}=\left|A\right|^{2}$$

b) Distributitve property

$$\vec{B}\bullet(\vec{A}+\vec{C})=\vec{B}\bullet\vec{A}+\vec{B}\bullet\vec{C}$$

3. vector product (cross product)

$$\vec{A}\times\vec{B} = \hat{n}\, AB\sin\theta_{AB}$$



a) anticommutative property

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$

b) Distributitve property

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$$

c) $\vec{A}\times\vec{A} = 0$

$$\hat{x}\times\hat{y} = \hat{z},\quad \hat{y}\times\hat{z} = \hat{x},$$
$$\hat{z}\times\hat{x} = \hat{y},$$

3. vector product (cross product)

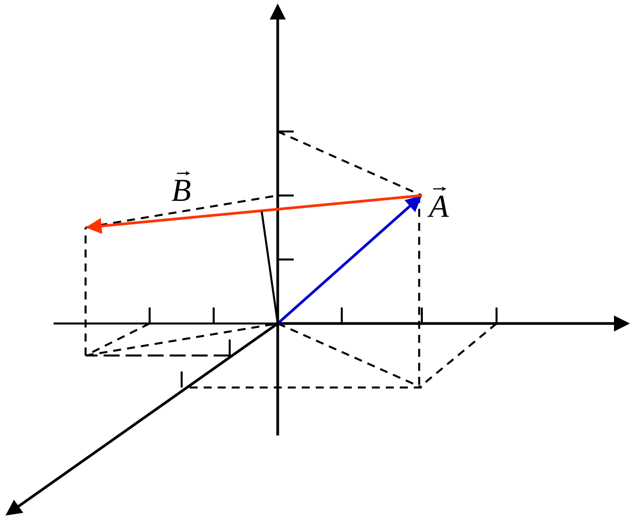
$$\vec{A}\times\vec{B} = (\hat{x}A_x + \hat{y}A_y + \hat{z}A_z)\times(\hat{x}B_x + \hat{y}B_y + \hat{z}B_z)$$

$$= \hat{x}(A_yB_z - A_zB_y) + \hat{y}(A_zB_x - A_xB_z) + \hat{z}(A_xB_y - A_yB_x)$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Example vectors and angles

In Cartesian coordinate, vector A is directed from origin to point P1(2,3,3), and vector B is directed from P1 to pint P2(1,-2,2). Find:
(a) Vector A, its magnitude |A|, and unit vector a
(b) the angle that A makes with the y-axis
(c) Vector B
(d) the angle between A and B
(e) perpendicular distance from origin to vector B



$$\vec{A} = \hat{x}2 + \hat{y}3 + \hat{z}3 \qquad |A| = \sqrt{2^2 + 3^2 + 3^2}$$

$$\hat{a} = \frac{\vec{A}}{|A|} = (\hat{x}2 + \hat{y}3 + \hat{z}3)/\sqrt{22}$$

$$\vec{A} \bullet \hat{y} = |A||\hat{y}|\cos(\theta_{Ay})$$

$$\vec{B} = \vec{P}_2 - \vec{P}_1 = \hat{x}(1-2) + \hat{y}(-2-3) + \hat{z}(2-3)$$

$$\vec{A} \bullet \vec{B} = |A||B|\cos(\theta_{AB})$$

$$d = |A|\sin(180^0 - \theta_{AB})$$

4. Scalar and vector triple product

a) scalar triple product

$$\vec{A}\bullet(\vec{B}\times\vec{C}) = \vec{B}\bullet(\vec{C}\times\vec{A}) = \vec{C}\bullet(\vec{A}\times\vec{B})$$

$$\vec{A}\bullet(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\bullet\vec{C}) - \vec{C}(\vec{A}\bullet\vec{B})\times)\times \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \vec{B}\bullet(\vec{C}\times\vec{A}) = \vec{C}\bullet(\vec{A}\times\vec{B})$$

b) vector triple product

$$\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\bullet\vec{C}) - \vec{C}(\vec{A}\bullet\vec{B})$$

Example vector triple product

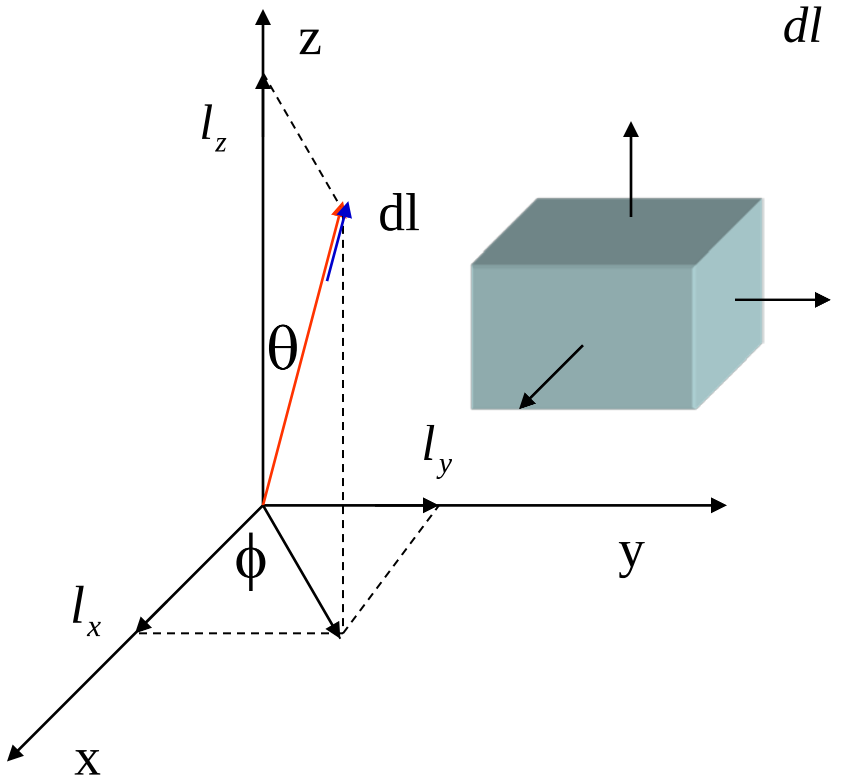
$$\vec{A} = \hat{x} - \hat{y} + \hat{z}2 \quad \vec{B} = \hat{y} + \hat{z} \quad \vec{C} = -2\hat{x} + \hat{z}3$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = ? \quad (\vec{A}\times\vec{B})\times\vec{C} = ?$$

$$\vec{B}\times\vec{C} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 1 & 1 \\ -2 & 0 & 3 \end{vmatrix} = \hat{x}3 - 2\hat{y} + 2\hat{z}$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = 3\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & -1 & 2 \\ -2 & -2 & 2 \end{vmatrix} = \hat{x}2 - 6\hat{y} + \hat{z}0$$

- Cartesian coordinate system



$$d\vec{l} = dl_x \hat{x} + dl_y \hat{y} + dl_z \hat{z} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

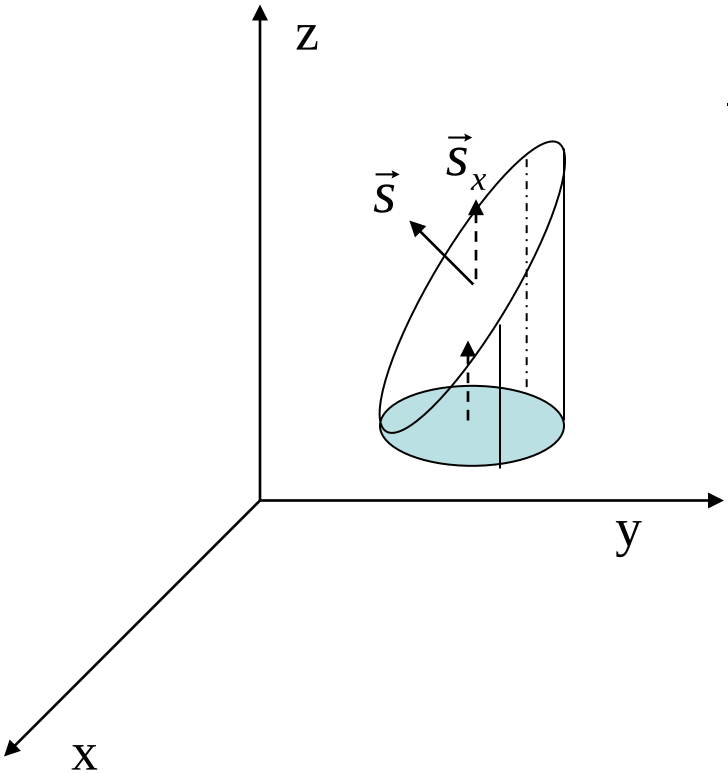
$$d\vec{s}_x = \hat{x}\,dl_y dl_z = \hat{x}\,dydz$$

$$d\vec{s}_y = \hat{y}\,dl_x dl_z = \hat{y}\,dxdz$$

$$d\vec{s}_z = \hat{z}\,dl_x dl_y = \hat{z}\,dxdy$$

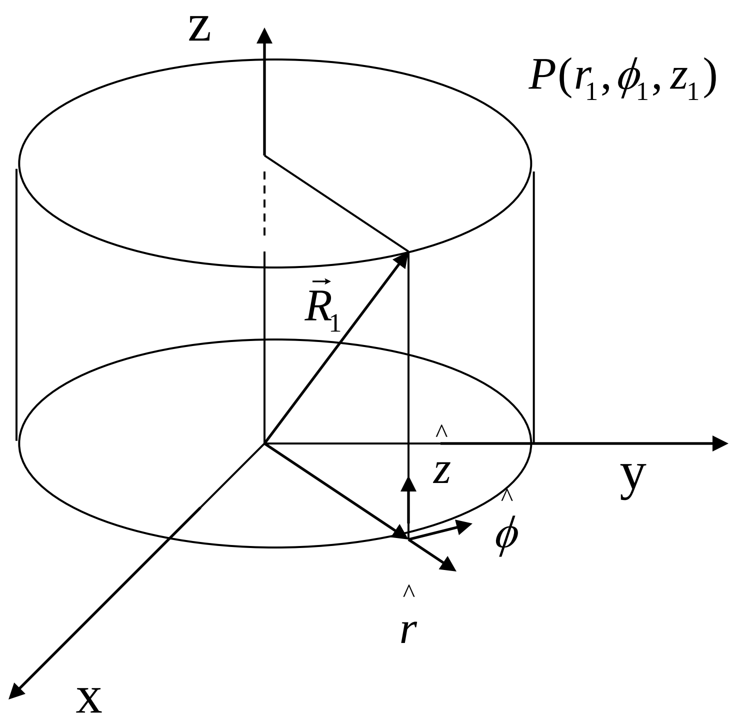
$$dv = dxdydz$$

• Cartesian coordinate system



directions of area

• Cylindrical coordinate system



$$\hat{r} \times \hat{\phi} = \hat{z} \qquad \hat{\phi} \times \hat{z} = \hat{r} \qquad \hat{z} \times \hat{r} = \hat{\phi}$$

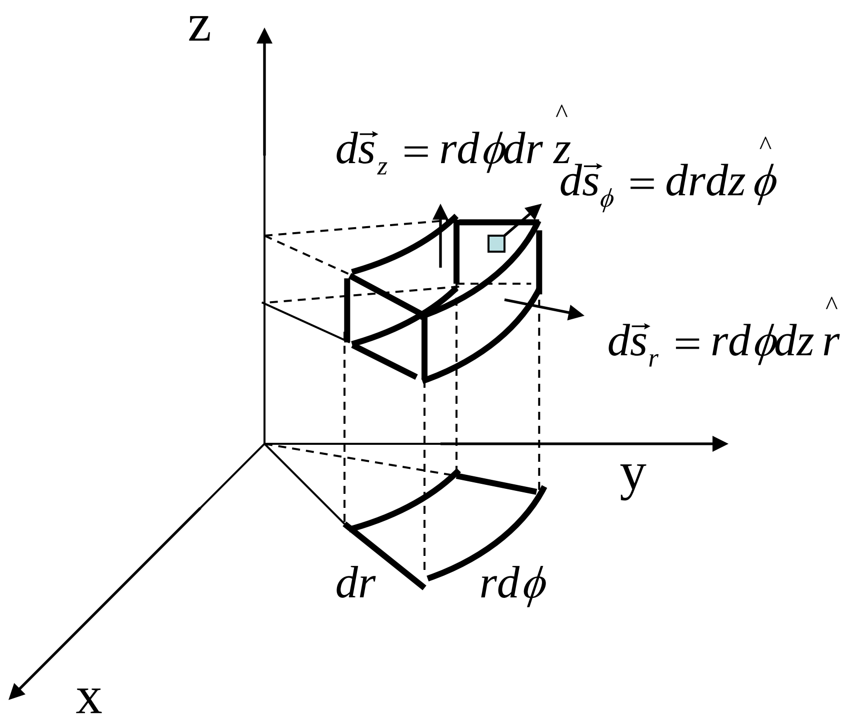
$$\hat{r} \times \hat{r} = 0 \qquad \hat{r} \bullet \hat{r} = 1$$

$$\vec{A} = \hat{r} A_r + \hat{\phi} A_\phi + \hat{z} A_z$$

$$\left|\vec{A}\right| = \sqrt{A_r^2 + A_\phi^2 + A_z^2}$$

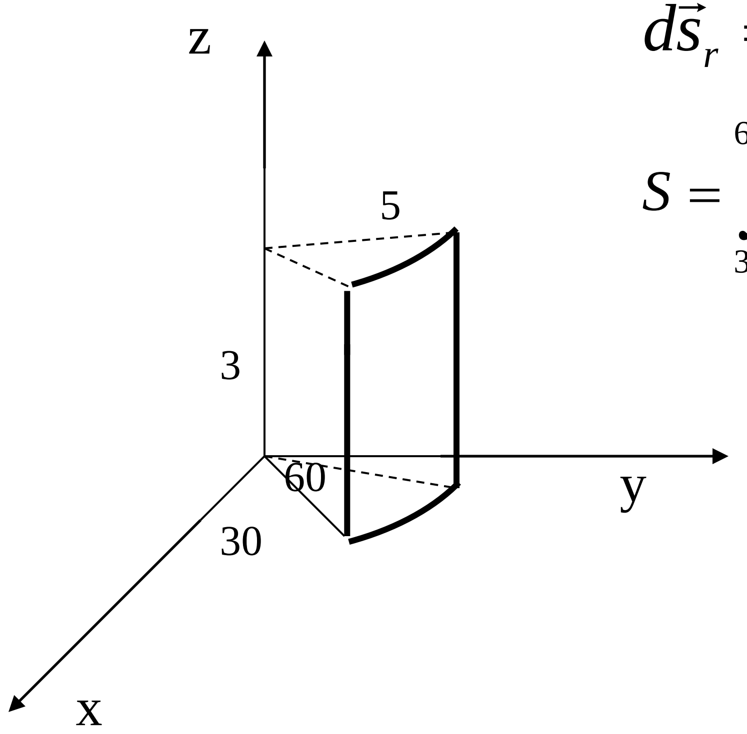
$$\vec{R}_1 = \hat{r} r_1 + \hat{z} z_1$$

• <u>the differential areas and volume</u>



$$dv = rd\phi drdz$$

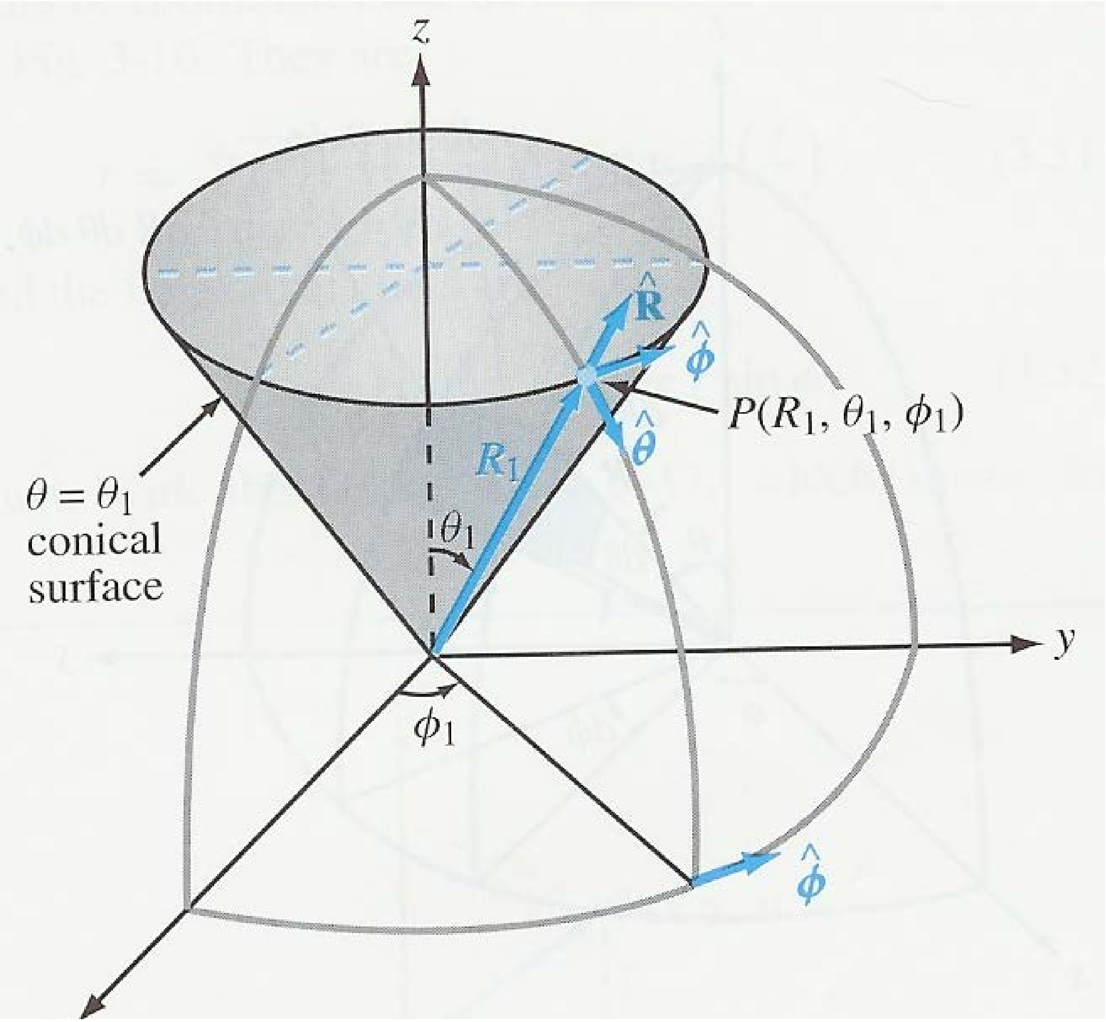
Example: cylindrical area



$$d\vec{s}_r = r\,d\phi\,dz\,\hat{r}$$

$$S = \int_{30}^{60} r\,d\phi \int_{0}^{3} dz = 5\cdot 3 \int_{\pi/6}^{\pi/3} d\phi$$

• Spherical coordinate system



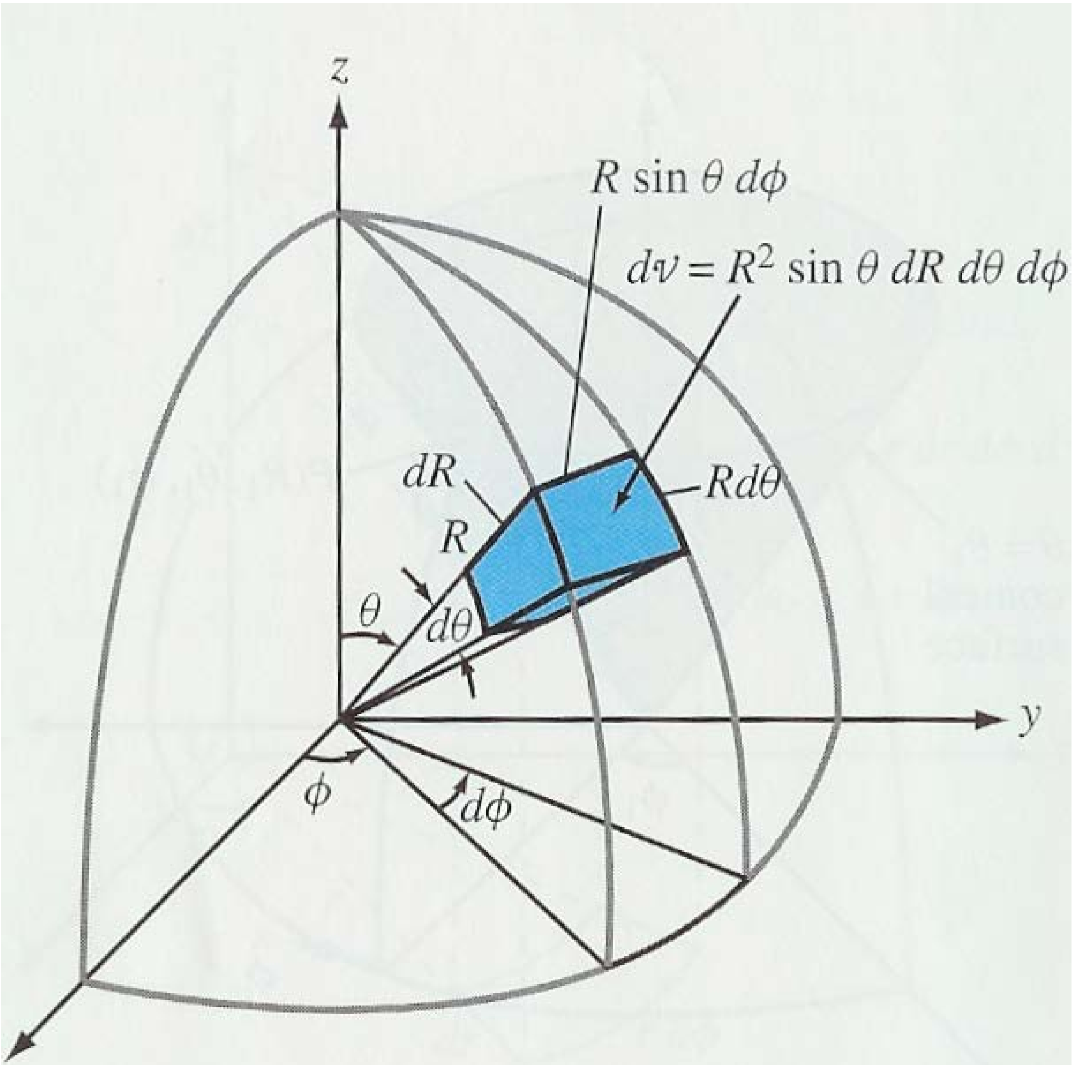
$$\hat{r} \times \hat{\theta} = \hat{\phi} \qquad \hat{\theta} \times \hat{\phi} = \hat{R} \qquad \hat{\phi} \times \hat{R} = \hat{\theta}$$

$$\hat{r} \times \hat{r} = 0 \qquad \hat{r} \bullet \hat{r} = 1$$

$$\vec{A} = \hat{R}\, A_R + \hat{\theta}\, A_\theta + \hat{\phi}\, A_\phi$$

$$\left|\vec{A}\right| = \sqrt{A_R^2 + A_\theta^2 + A_\phi^2}$$

- differential volume in Spherical coordinate system



$$d\vec{l} = \hat{R}\, dl_R + \hat{\theta}\, dl_\theta + \hat{\phi}\, dl_\phi$$

$$= \hat{R}\, dR + \hat{\theta}\, R d\theta + \hat{\phi}\, R \sin\theta d\phi$$

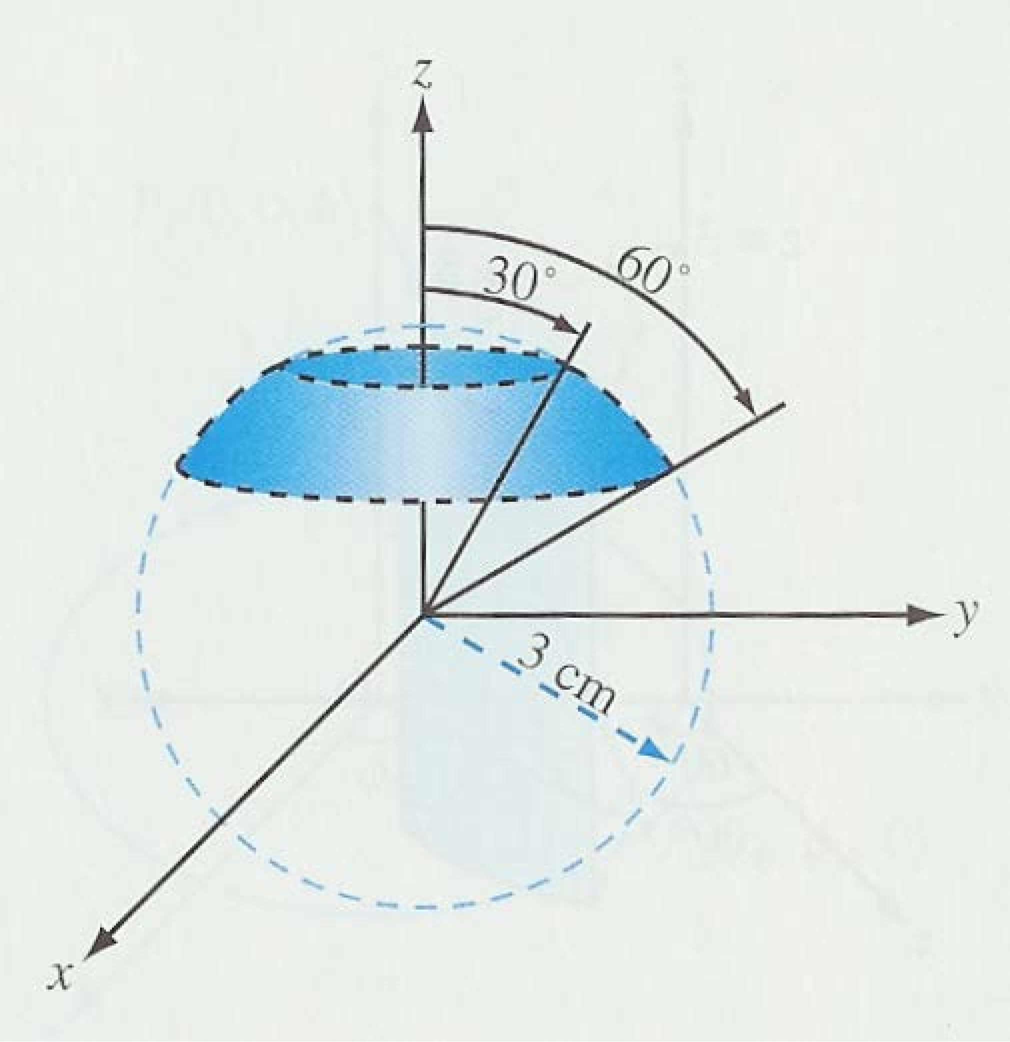
$$d\vec{s}_R = \hat{R}\, dl_\theta dl_\phi = \hat{R}\, R^2 \sin\theta d\theta d\phi$$

$$d\vec{s}_\theta = \hat{\theta}\, dl_R dl_\phi = \hat{\theta}\, R \sin\theta dR d\phi$$

$$d\vec{s}_\phi = \hat{\phi}\, dl_\theta dl_R = \hat{R}\, R d\theta d\phi$$

$$dv = dR dl_\theta dl_\phi = R^2 \sin\theta dR d\theta d\phi$$

- Examples



(1) Find the area of the strip

$$S = \int d\vec{S}_R = \int_{30}^{60} \int_{0}^{360} R^2 \sin\theta d\theta d\phi$$

(2) A sphere of radius 2cm contains a volume charge density $\rho_v = 4\cos^2\theta$

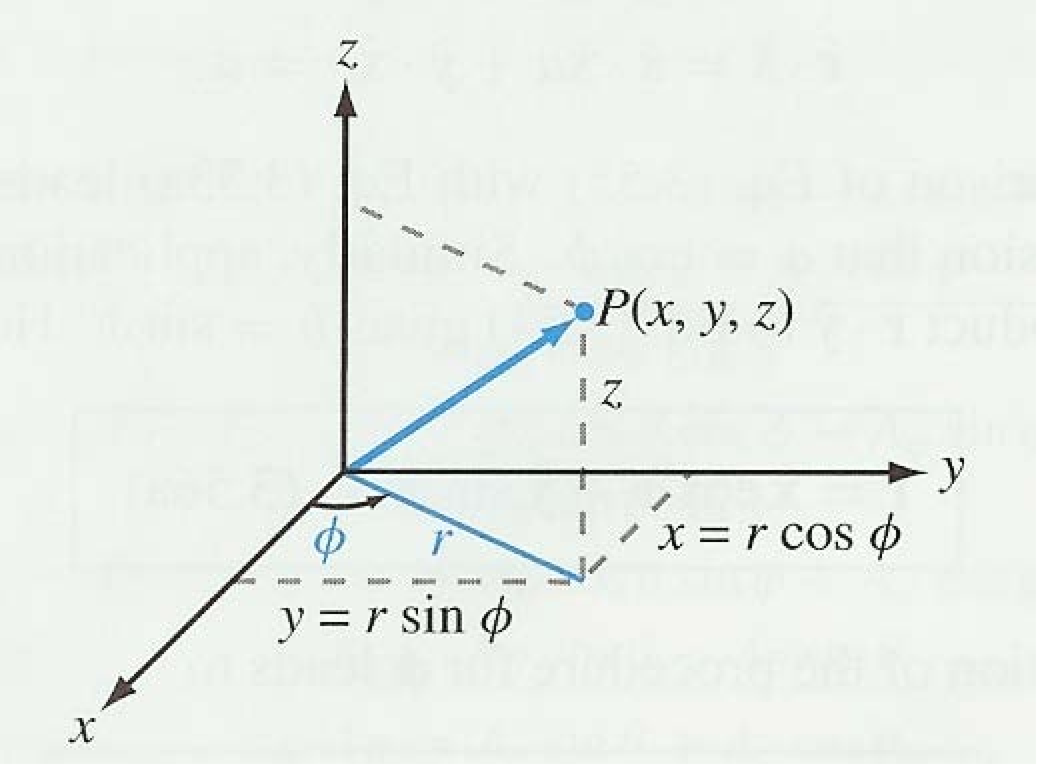
Find the total charge contained in the sphere

$$Q = \iiint dv\rho_v = \int_0^{2/100} dR \int_0^{180} d\theta \int_0^{260} d\phi \rho_v (R^2 \sin\theta)$$

$$= \int_0^{2/100} dRR^2 \int_0^{180} d\theta \int_0^{260} d\phi (4\cos^2\theta)(\sin\theta)$$

$$= \int_0^{2/100} dRR^2 \int_0^{180} d\theta (4\cos^2\theta)(\sin\theta) \int_0^{260} d\phi$$

- Cartesian to cylindrical transformation



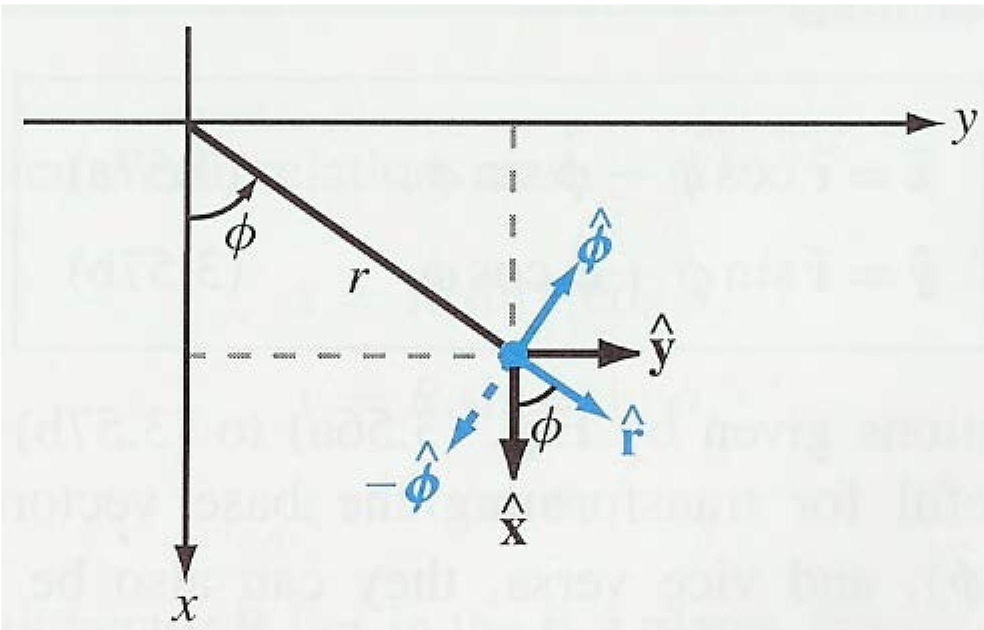
$$r = \sqrt{x^2 + y^2},$$

$$\phi = \tan^{-1}(\frac{y}{x}),$$

$$x = r\cos(\phi),$$

$$y = r\sin(\phi),$$

• Cartesian to cylindrical transformation



$$\hat{r}\bullet\hat{x}=\cos(\phi),\quad \hat{r}\bullet\hat{y}=\sin(\phi),$$

$$\hat{\phi}\bullet\hat{x}=-\sin(\phi),\quad \hat{\phi}\bullet\hat{y}=\cos(\phi),$$

$$\hat{r}\bullet\hat{x}=(a\hat{x}+b\hat{y})\bullet\hat{x}=a=\cos(\phi),$$

$$\hat{r}\bullet\hat{y}=(a\hat{x}+b\hat{y})\bullet\hat{y}=b=\sin(\phi),$$

$$\boxed{\hat{r}=\hat{x}\cos\phi+\hat{y}\sin\phi,}$$

$$\boxed{\hat{\phi}=-\hat{x}\sin\phi+\hat{y}\cos\phi,}$$

$$\boxed{\hat{x}=\hat{r}\cos\phi-\hat{\phi}\sin\phi,}$$

$$\boxed{\hat{y}=\hat{r}\sin\phi+\hat{\phi}\cos\phi,}$$

• Cartesian to cylindrical transformation

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z},$$

$$\hat{r} \bullet \hat{x} = \cos(\phi), \quad \hat{r} \bullet \hat{y} = \sin(\phi),$$

$$\longrightarrow \quad \vec{A} = A_r \hat{r} + A_\phi \hat{\phi} + A_z \hat{z},$$

$$\hat{\phi} \bullet \hat{x} = -\sin(\phi), \quad \hat{\phi} \bullet \hat{y} = \cos(\phi),$$

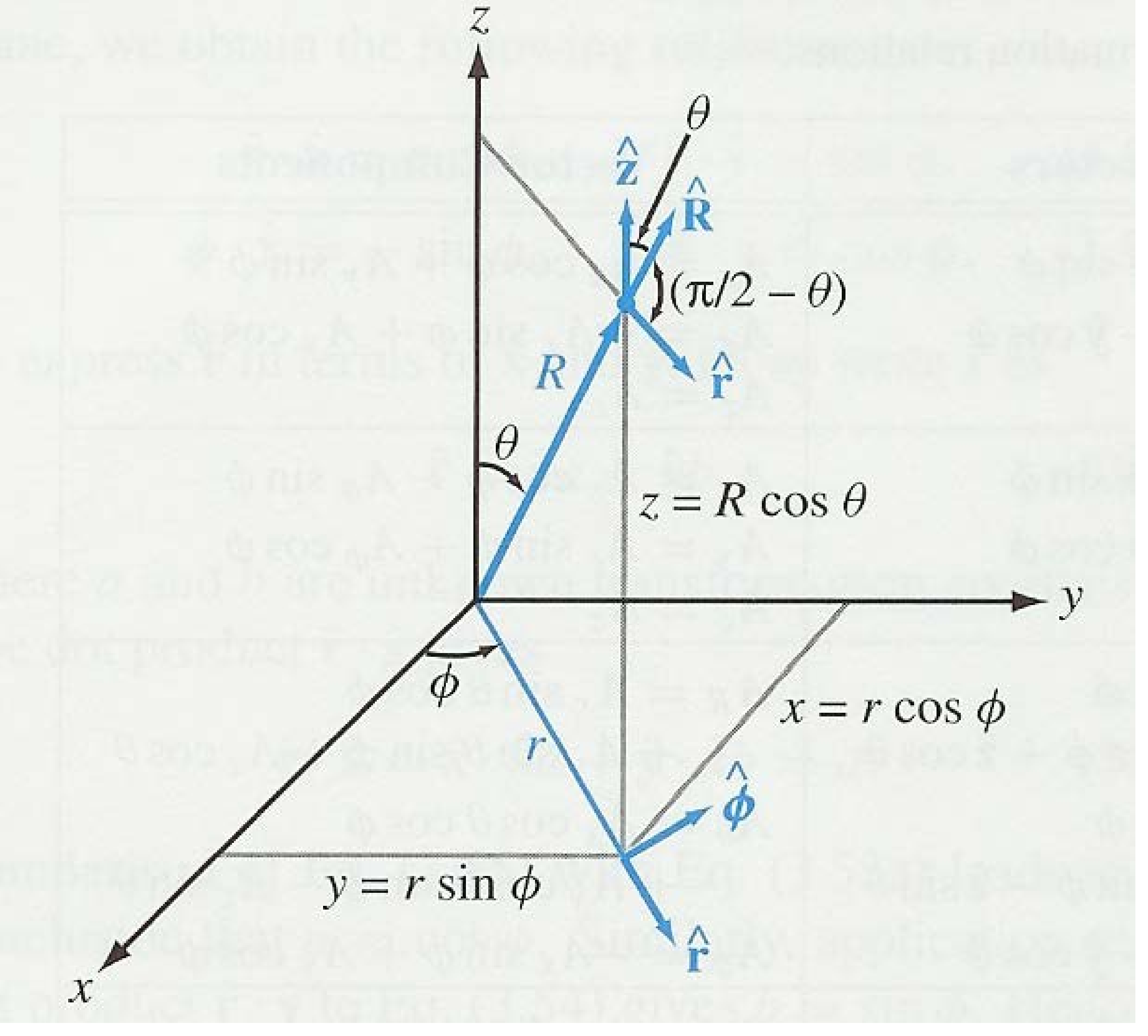
$$\boxed{A_r = A_x \cos\phi + A_y \sin\phi,}$$

$$\boxed{A_\phi = -A_x \sin\phi + A_y \cos\phi,}$$

$$\boxed{A_x = A_r \cos\phi - A_\phi \sin\phi,}$$

$$\boxed{A_y = A_r \sin\phi + A_\phi \cos\phi,}$$

• Cartesian to Spherical transformation



$$r = \sqrt{x^2 + y^2 + z^2},$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right),$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right),$$

$$x = R\sin\theta\cos\phi,$$

$$y = R\sin\theta\sin\phi,$$

$$z = R\cos\theta,$$

- Cartesian to Spherical transformation

$$\hat{R} = \hat{r}a + \hat{z}b,$$

$$\hat{R} \bullet \hat{r} = a = \cos\theta, \qquad \hat{r} = \hat{x}\cos\phi + \hat{y}\sin\phi,$$

$$\hat{R} \bullet \hat{z} = b = \sin\theta,$$

$$\boxed{\hat{R} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta,}$$

$$\hat{\theta} = \hat{r}c + \hat{z}d, \qquad \hat{\theta} \bullet \hat{r} = c = \cos\theta, \quad \hat{\theta} \bullet \hat{r} = c = -\sin\theta,$$

$$\boxed{\hat{\theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta,}$$

$$\boxed{\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi,}$$

- Cartesian to Spherical transformation

$$\hat{x} = \hat{R}c + \hat{\theta}d + \hat{\phi}e,$$

$$\hat{R} \bullet \hat{x} = \sin\theta\cos\phi, \qquad \hat{\theta} \bullet \hat{x} = \cos\theta\cos\phi, \qquad \hat{\phi} \bullet \hat{x} = e = -\sin\phi,$$

$$\hat{R} \bullet \hat{z} = b = \sin\theta,$$

$$\hat{r} = \hat{x}\cos\phi + \hat{y}\sin\phi,$$

$$\boxed{\hat{x} = \hat{R}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi,}$$

$$\hat{\theta} = \hat{r}c + \hat{z}d, \qquad \hat{\theta} \bullet \hat{r} = c = \cos\theta, \quad \hat{\theta} \bullet \hat{r} = c = -\sin\theta,$$

$$\boxed{\hat{y} = \hat{R}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi,}$$

$$\boxed{\hat{z} = \hat{R}\cos\theta - \hat{\theta}\sin\theta,}$$

• Distance between two points:

$$d_{12} = |R_{12}| = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{1/2},$$

$$d_{12} = |R_{12}| = \left[(r_2 \cos\phi_2 - r_1 \cos\phi_1)^2 + (r_2 \sin\phi_2 - r_1 \sin\phi_1)^2 + (z_2 - z_1)^2\right]^{1/2},$$

$$d_{12} = |R_{12}| = \left[(R_2 \sin\theta_2 \cos\phi_2 - R_1 \sin\theta_1 \cos\phi_1)^2 + (R_2 \sin\theta_2 \sin\phi_2 - R_1 \sin\theta_1 \sin\phi_1)^2 + (R_2 \cos\theta_2 - R_1 \cos\theta_1)^2\right]^{1/}$$

$$= \left\{R_2^2 + R_1^2 - 2R_1R_2[\cos\theta_2 \cos\theta_1 + \sin\theta_2 \sin\theta_1 \cos(\phi_2 - \phi_1)\right\}^{1/2},$$

Gradient in Cartesian Coordinates

Gradient: differential change of a scalar

$$dT(x,y,z) = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz,$$

$$= \frac{\partial T}{\partial x}\hat{x} \bullet d\vec{l} + \frac{\partial T}{\partial y}\hat{y} \bullet d\vec{l} + \frac{\partial T}{\partial z}\hat{z} \bullet d\vec{l}$$

$$= (\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}) \bullet d\vec{l},$$

$$= \nabla T \bullet d\vec{l},$$

$$\nabla T \equiv (\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}),$$

$$\Delta T = \int dT(x,y,z) = \int \nabla T \bullet d\vec{l}$$

The direction of ∇T is along the maximum increase of T.

Example of Gradient in Cartesian Coordinates

Find the directional derivative of $T = x^2 + y^2 z,$ along the direction $\hat{x}2 + \hat{y}3 - \hat{z}2$

and evaluate it at (1, -1,2).

$$\frac{dT}{dl} = \nabla T \bullet \hat{a}_l ,$$

$$\nabla T = (\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z})(x^2 + y^2 z),$$

Gradient operator in cylindrical Coordinates

$$\nabla T = \hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial z},$$

$$\hat{x} = \hat{r}\cos\phi - \hat{\phi}\sin\phi,$$

$$\hat{y} = \hat{r}\sin\phi + \hat{\phi}\cos\phi,$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial T}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial x},$$

$$= \frac{\partial T}{\partial r}\cos\phi + \frac{\partial T}{\partial \phi}(-\frac{1}{r}\sin\phi),$$

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial T}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial y},$$

$$= \frac{\partial T}{\partial r}\sin\phi + \frac{\partial T}{\partial \phi}(\frac{1}{r}\cos\phi),$$

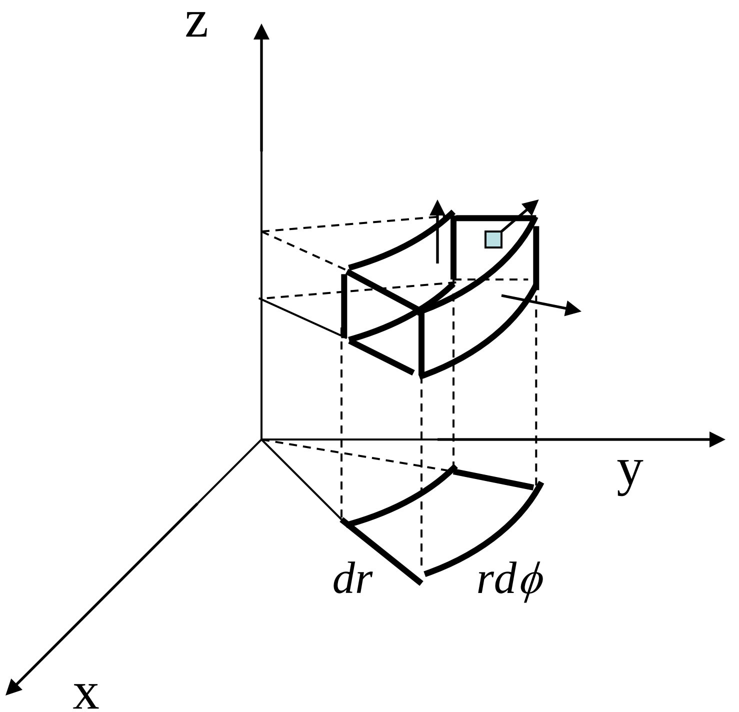
$$\nabla T = \hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial z}$$

$$= \hat{r}(,\frac{\partial T}{\partial r}\cos^2\phi - \frac{\partial T}{\partial \phi}\frac{1}{r}\cos\phi\sin\phi + \frac{\partial T}{\partial r}\sin^2\phi + \frac{\partial T}{\partial \phi}\frac{1}{r}\cos\phi\sin\phi)$$

$$+ \hat{\phi}(-\frac{\partial T}{\partial r}\sin\phi\cos\phi + \frac{\partial T}{\partial \phi}\frac{1}{r}\sin^2\phi + \frac{\partial T}{\partial \phi}\frac{1}{r}\cos^2\phi + \frac{\partial T}{\partial r}\sin\phi\cos\phi)$$

$$= \hat{r}\frac{\partial T}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial T}{\partial \phi} + \hat{z}\frac{\partial T}{\partial z}$$

Gradient operator in cylindrical Coordinates



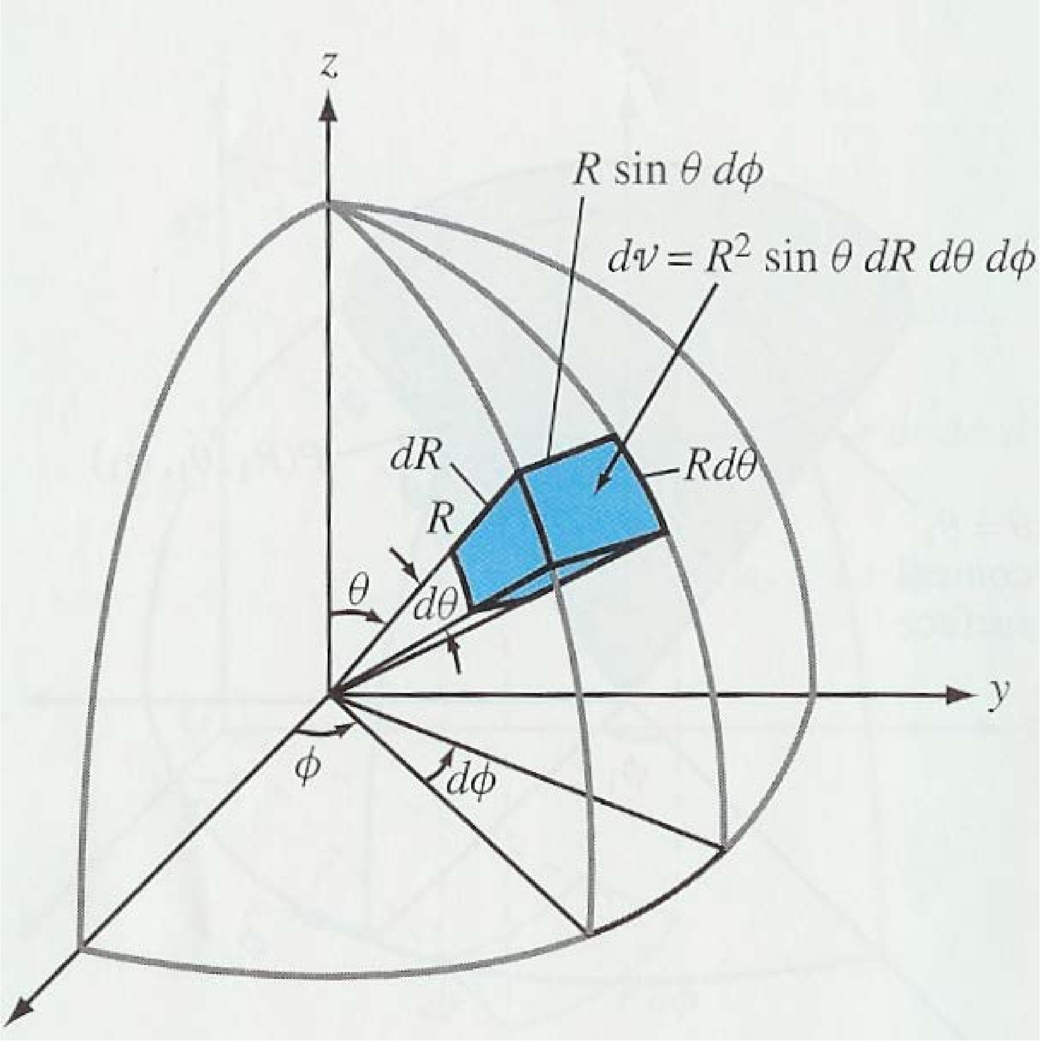
$$\nabla T = \hat{r}\frac{\partial T}{\partial l_r} + \hat{\phi}\frac{\partial T}{\partial l_\phi} + \hat{z}\frac{\partial T}{\partial l_z}$$

$$l_r = dr, \quad l_\phi = rd\phi, \quad l_z = dz,$$

$$\nabla T = \hat{r}\frac{\partial T}{\partial l_r} + \hat{\phi}\frac{\partial T}{\partial l_\phi} + \hat{z}\frac{\partial T}{\partial l_z}$$

$$= \hat{r}\frac{\partial T}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial T}{\partial \phi} + \hat{z}\frac{\partial T}{\partial z}$$

Gradient operator in Spherical Coordinates



$$\nabla T = \hat{R}\frac{\partial T}{\partial l_R} + \hat{\theta}\frac{\partial T}{\partial l_\theta} + \hat{\phi}\frac{\partial T}{\partial l_\phi},$$

$$l_R = dR, \quad l_\theta = Rd\theta, \quad l_\phi = R\sin\theta d\phi,$$

$$\nabla T = \hat{R}\frac{\partial T}{\partial l_R} + \hat{\theta}\frac{\partial T}{\partial l_\theta} + \hat{\phi}\frac{\partial T}{\partial l_\phi}$$

$$= \hat{R}\frac{\partial T}{\partial R} + \hat{\theta}\frac{1}{R}\frac{\partial T}{\partial \theta} + \hat{\phi}\frac{1}{R\sin\theta}\frac{\partial T}{\partial \phi}$$

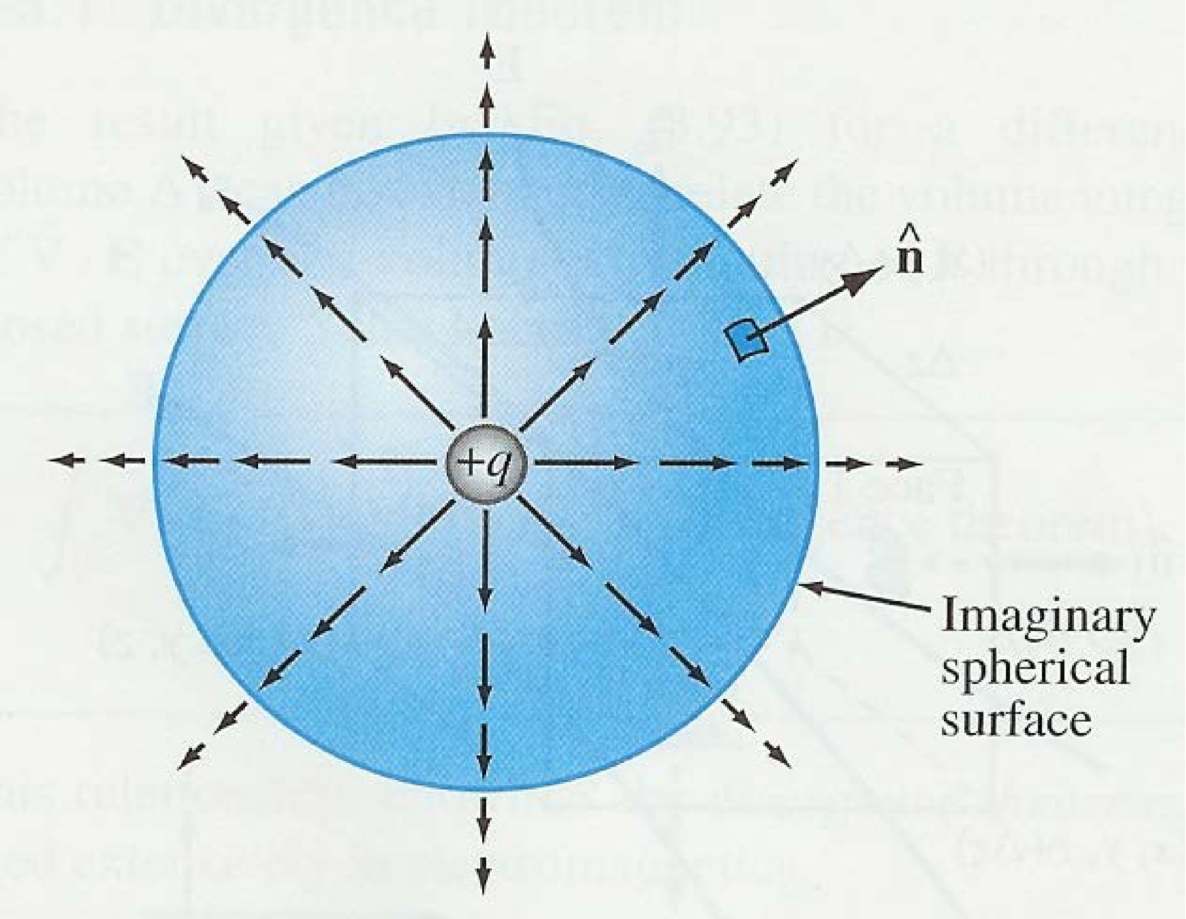
Properties of the Gradient operator

$$\nabla(U+V)=\nabla U+\nabla V,$$

$$\nabla(UV)=U\nabla V+V\nabla U,$$

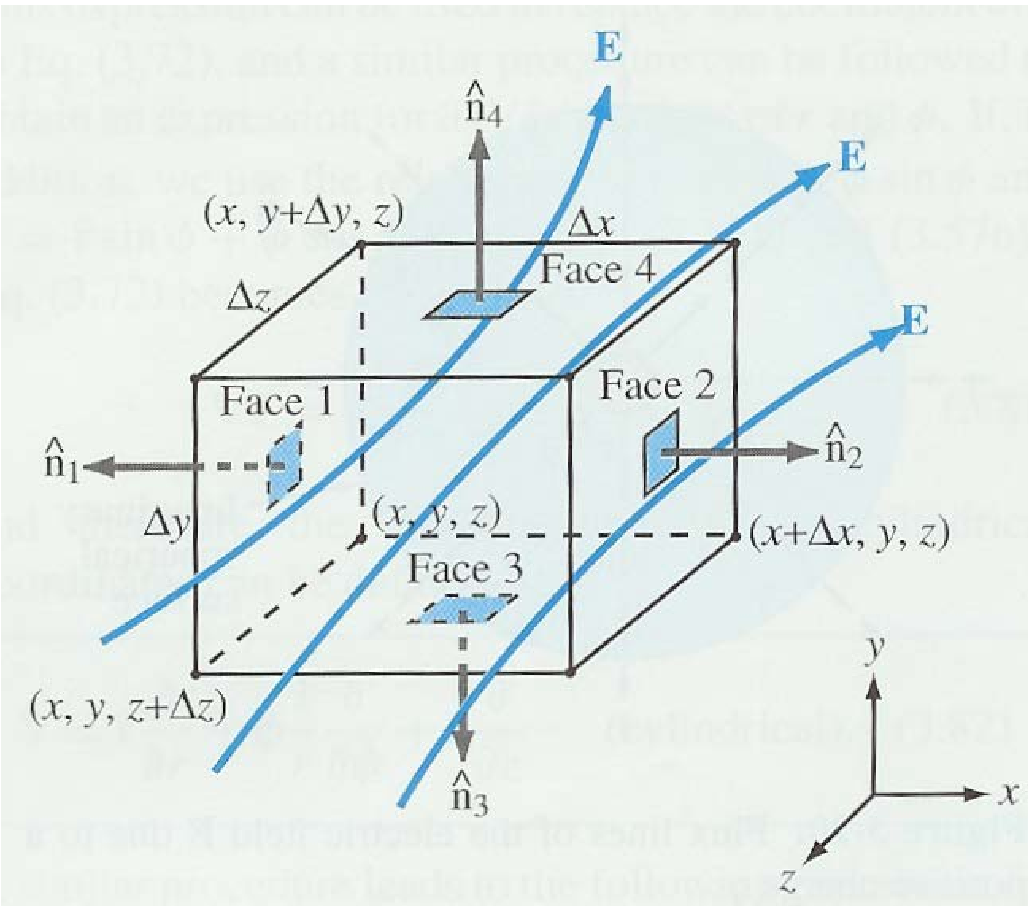
$$\nabla(V^{n})=nV^{n-1}\nabla V,$$

Flux in Cartesian Coordinates



$$Total \quad flux = \oint \vec{E} \bullet d\vec{s},$$

Flux in Cartesian Coordinates



$$F_1 = \int_{face1} \vec{E} \bullet \vec{n}_1 ds,$$

$$= \int_{face1} (\hat{x}E_x + \hat{y}E_y + \hat{z}E_z) \bullet (-\hat{x}) dydz$$

$$= -E_x(1)\Delta y \Delta z,$$

$$F_2 = \int_{face2} \vec{E} \bullet \vec{n}_2 ds,$$

$$= \int_{face2} (\hat{x}E_x + \hat{y}E_y + \hat{z}E_z) \bullet (\hat{x}) dydz$$

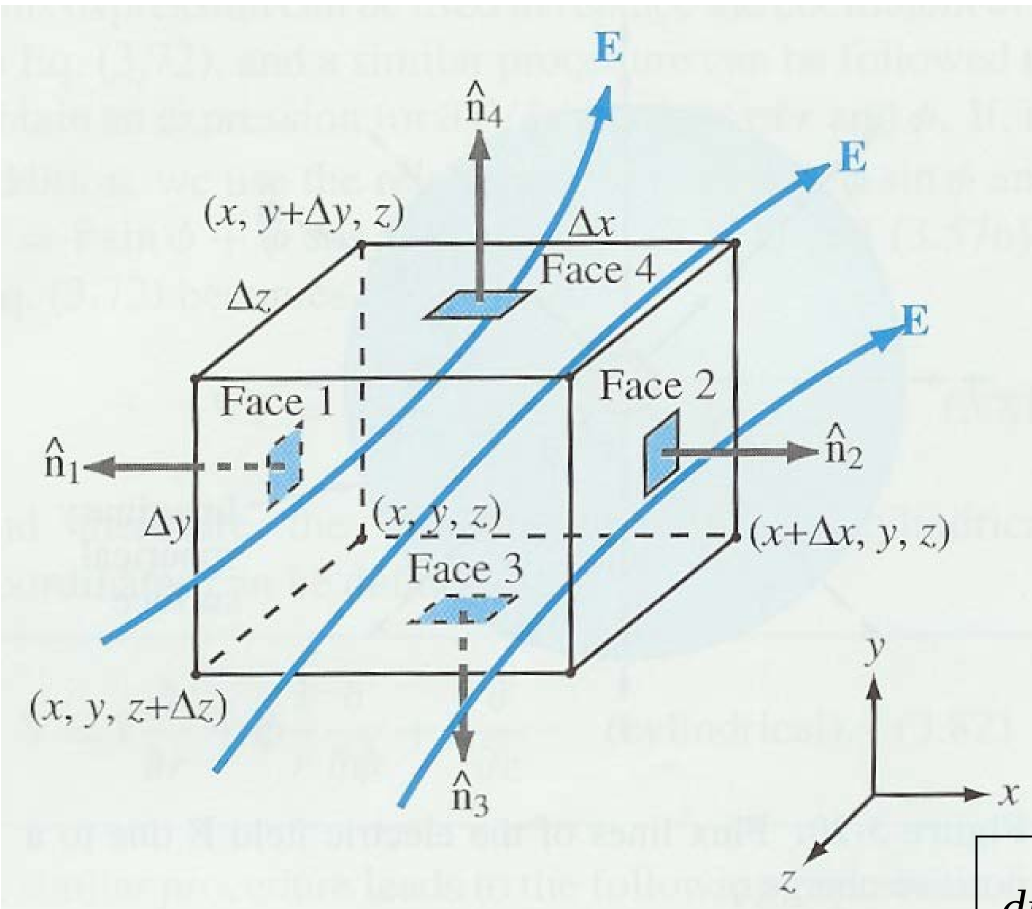
$$= E_x(2)\Delta y \Delta z,$$

$$F_1 + F_2 = [E_x(2) - E_x(1)]\Delta y \Delta z$$

$$= \frac{[E_x(2) - E_x(1)]}{\Delta x} \Delta x \Delta y \Delta z$$

$$= \frac{\partial E_x}{\partial x} \Delta x \Delta y \Delta z,$$

Definition of divergence in Cartesian Coordinates



$$F_1 + F_2 = [E_x(2) - E_x(1)]\Delta y \Delta z$$

$$= \frac{[E_x(2) - E_x(1)]}{\Delta x}\Delta x \Delta y \Delta z$$

$$= \frac{\partial E_x}{\partial x}\Delta x \Delta y \Delta z,$$

$$F_3 + F_4 = \frac{\partial E_y}{\partial y}\Delta x \Delta y \Delta z,$$

$$F_5 + F_6 = \frac{\partial E_z}{\partial z}\Delta x \Delta y \Delta z,$$

$$\oint_S \vec{E} \bullet d\vec{s} = (\frac{\partial E_x}{\partial z} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z})\Delta x \Delta y \Delta z,$$

$$div\vec{E} \equiv \nabla \bullet \vec{E} = (\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}),$$

Properties of divergence

$$\nabla \bullet (\vec{E}_1 + \vec{E}_2) = \nabla \bullet \vec{E}_1 + \nabla \bullet \vec{E}_2,$$

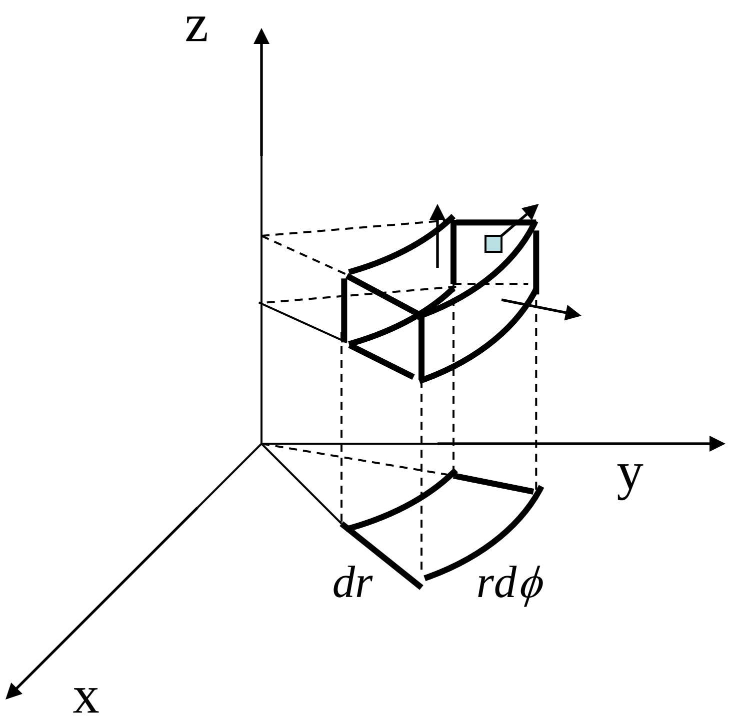
If $\nabla \bullet \vec{E}_1 = 0,$ No net flux on any closed surface.

Divergence theorem

$$\oint_S \vec{E} \bullet d\vec{s} = (\frac{\partial E_x}{\partial z} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z})\Delta x \Delta y \Delta z$$

$$= \int_v \nabla \bullet \vec{E} dxdydz,$$

Divergence in Cylindrical Coordinates



$$F_1 = \int_{face1} \vec{E} \bullet \vec{n}_r ds_r,$$

$$= \int_{face1} (\hat{r}E_r + \hat{\phi}E_\phi + \hat{z}E_z) \bullet (-\hat{r}) r\Delta\phi\Delta z$$

$$= -E_r(1) r\Delta\phi\Delta z,$$

$$F_2 = \int_{face2} \vec{E} \bullet \vec{n}_r ds_r,$$

$$= \int_{face1} (\hat{r}E_r + \hat{\phi}E_\phi + \hat{z}E_z) \bullet (-\hat{r}) r\Delta\phi\Delta z$$

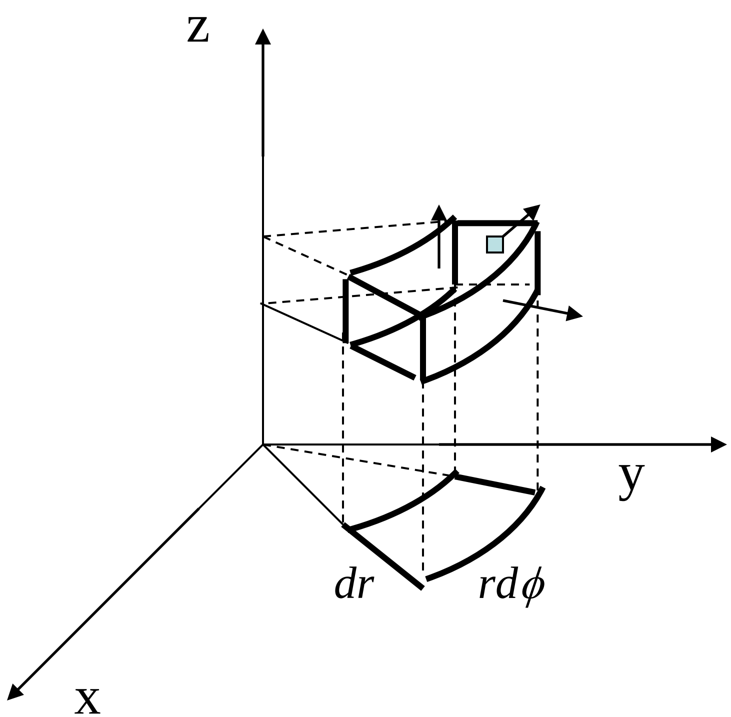
$$= E_r(2) r\Delta\phi\Delta z,$$

$$F_1 + F_2 = [E_r(2) - E_r(1)] r\Delta\phi\Delta z$$

$$= \frac{[rE_r(2) - rE_r(1)]}{\Delta r} \Delta r\Delta\phi\Delta z$$

$$= \frac{\partial}{\partial r}(rE_r) \Delta r\Delta\phi\Delta z,$$

Divergence in Cylindrical Coordinates



$$F_1 + F_2 = [E_r(2) - E_r(1)]r\Delta\phi\Delta z$$

$$= \frac{[rE_r(2) - rE_r(1)]}{\Delta r}\Delta r\Delta\phi\Delta z$$

$$= \frac{\partial}{\partial r}(rE_r)\Delta r\Delta\phi\Delta z,$$

$$F_3 + F_4 = [E_\phi(2) - E_\phi(1)]\Delta r\Delta z$$

$$= \frac{[E_\phi(2) - E_\phi(1)]}{\Delta\phi}\Delta r\Delta\phi\Delta z$$

$$= \frac{\partial}{\partial\phi}(E_\phi)\Delta r\Delta\phi\Delta z,$$

$$F_5 + F_6 = [E_z(2) - E_z(1)]r\Delta\phi\Delta r$$

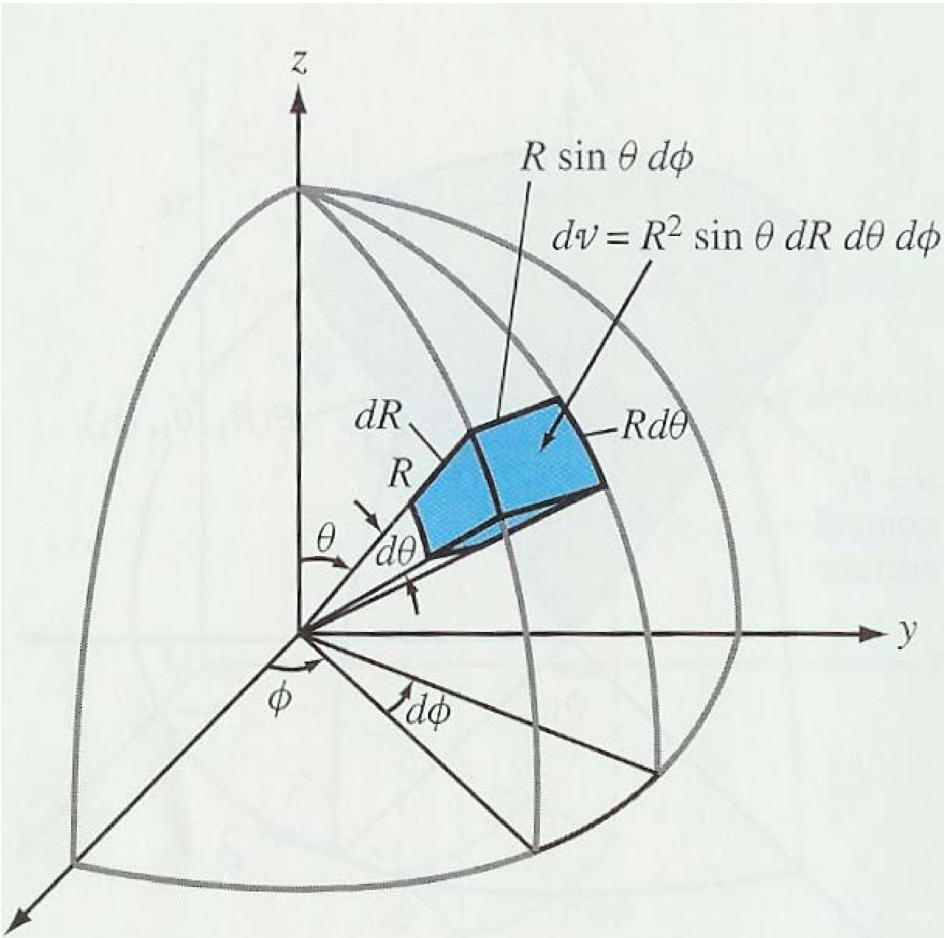
$$= \frac{[E_z(2) - E_z(1)]}{\Delta z}r\Delta r\Delta\phi\Delta z$$

$$= \frac{\partial}{\partial z}(E_z)r\Delta r\Delta\phi\Delta z,$$

$$\oint_S \vec{E} \bullet d\vec{s} = \left(\frac{\partial(rE_r)}{r\partial r} + \frac{\partial E_\phi}{r\partial\phi} + \frac{\partial E_z}{\partial z}\right)r\Delta r\Delta\phi\Delta z$$

$$= \nabla \bullet \vec{E}\Delta v,$$

Divergence in Spherical Coordinates



$$F_1 = \int_{face1} \vec{E} \bullet \vec{n}_R ds_R,$$

$$= \int_{face1} (\hat{R}E_R + \hat{\theta}E_\theta + \hat{\phi}E_\phi) \bullet (-\hat{R}) R\sin\theta\Delta\phi R\Delta\theta$$

$$= -E_R(1)R^2 \sin\theta\Delta\phi\Delta\theta,$$

$$F_2 = \int_{face2} \vec{E} \bullet \vec{n}_R ds_R,$$

$$= \int_{face2} (\hat{R}E_R + \hat{\theta}E_\theta + \hat{\phi}E_\phi) \bullet (\hat{R}) R\sin\theta\Delta\phi R\Delta\theta$$

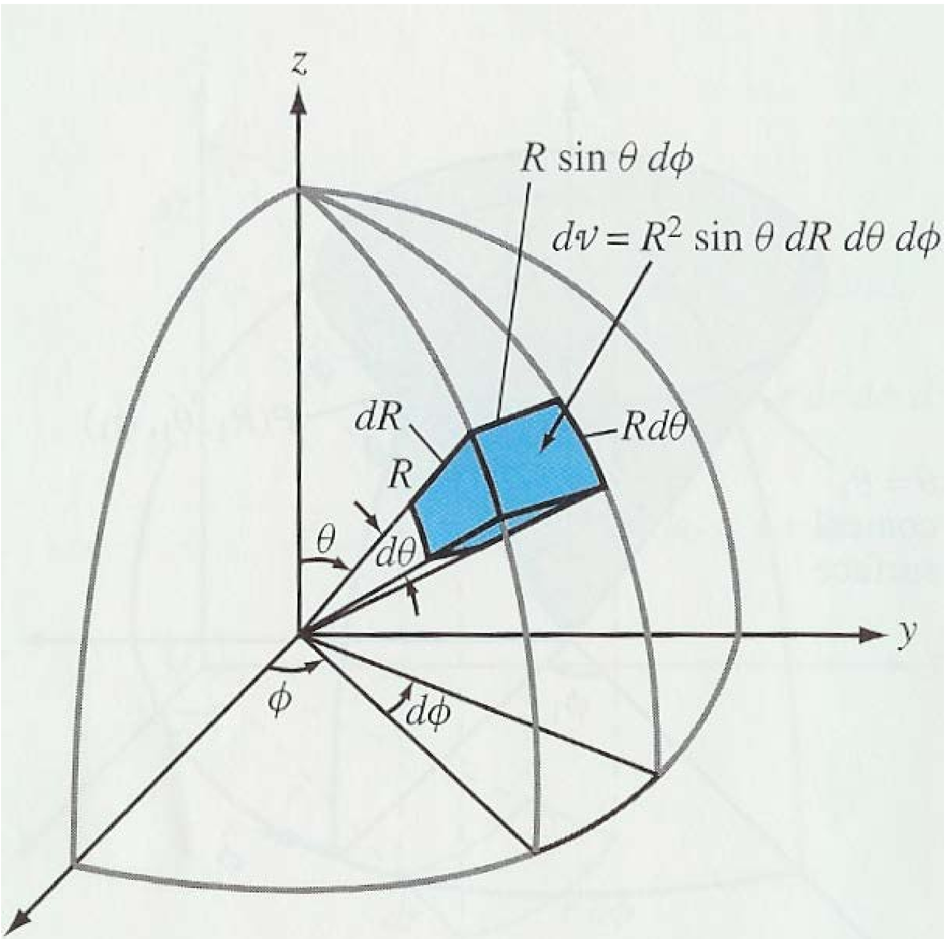
$$= -E_R(2)R^2 \sin\theta\Delta\phi\Delta\theta,$$

$$F_1 + F_2 = [E_R(2) - E_R(1)]R^2 \sin\theta\Delta\theta\Delta\phi$$

$$= \frac{[R^2 E_R(2) - R^2 E_R(1)]}{\Delta R} \sin\theta\Delta R\Delta\phi\Delta\theta$$

$$= \frac{\partial}{\partial R}(R^2 E_R) \sin\theta\Delta R\Delta\phi\Delta\theta,$$

Divergence in Spherical Coordinates



$$F_1 + F_2 = [E_R(2) - E_R(1)]R^2 \sin\theta \Delta\theta \Delta\phi$$

$$= \frac{[R^2 E_R(2) - R^2 E_R(1)]}{\Delta R} \sin\theta \Delta R \Delta\phi \Delta\theta$$

$$= \frac{\partial}{\partial R}(R^2 E_R) \sin\theta \Delta R \Delta\phi \Delta\theta,$$

$$F_3 + F_4 = [E_\theta(2) - E_\theta(1)]R \sin\theta \Delta R \Delta\phi$$

$$= \frac{[\sin\theta E_\theta(2) - \sin\theta E_\theta(1)]}{\Delta\theta} R \Delta R \Delta\phi \Delta\theta$$

$$= \frac{\partial}{\partial\theta}(\sin\theta E_R) R \Delta R \Delta\phi \Delta\theta,$$

$$F_5 + F_6 = [E_\phi(2) - E_\phi(1)]R \Delta\theta \Delta R$$

$$= \frac{[E_\phi(2) - E_\phi(1)]}{\Delta\phi} R \Delta R \Delta\phi \Delta\theta$$

$$= \frac{\partial}{\partial\phi}(E_\phi) \Delta R \Delta\phi \Delta\theta,$$

Divergence in Spherical Coordinates

$$F_1 + F_2 = \frac{\partial}{\partial R}(R^2 E_R)\sin\theta \Delta R \Delta\phi \Delta\theta, \qquad F_3 + F_4 = \frac{\partial}{\partial \theta}(\sin\theta E_R) R \Delta R \Delta\phi \Delta\theta,$$

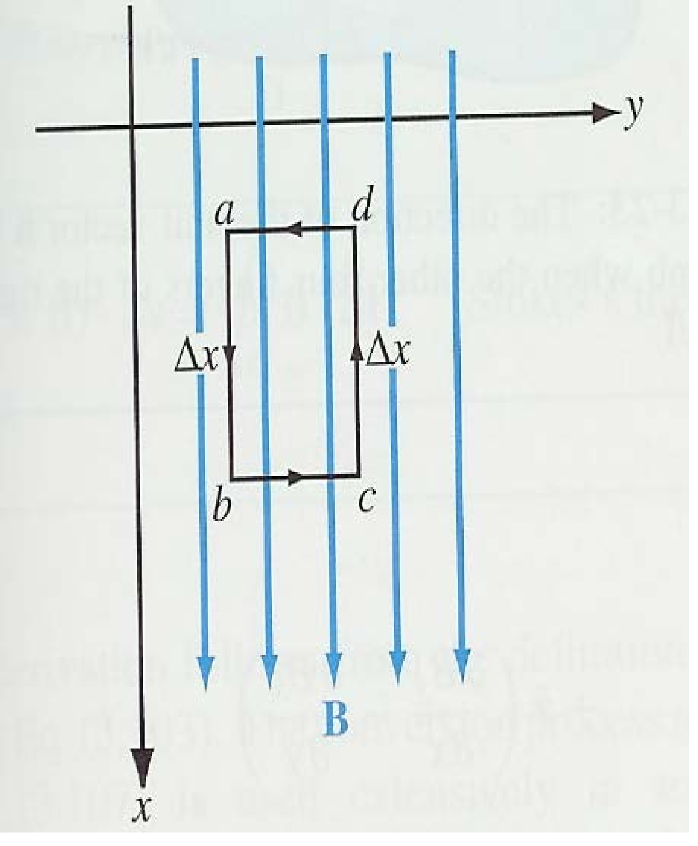
$$F_5 + F_6 = \frac{\partial}{\partial \phi}(E_\phi) R \Delta R \Delta\phi \Delta\theta,$$

$$\oint_S \vec{E} \bullet d\vec{s} = [\frac{\partial}{\partial R}(R^2 E_R) + R\frac{\partial}{\partial \theta}(\sin\theta E_\theta) + R\frac{\partial}{\partial \phi}(E_\phi)] \Delta R \Delta\phi \Delta\theta,$$

$$= [\frac{1}{R^2}\frac{\partial}{\partial R}(R^2 E_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta E_\theta) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \phi}(E_\phi)] R^2 \sin\theta \Delta R \Delta\phi \Delta\theta$$

$$= \nabla \bullet \vec{E} \Delta v$$

Circulation of a Vector



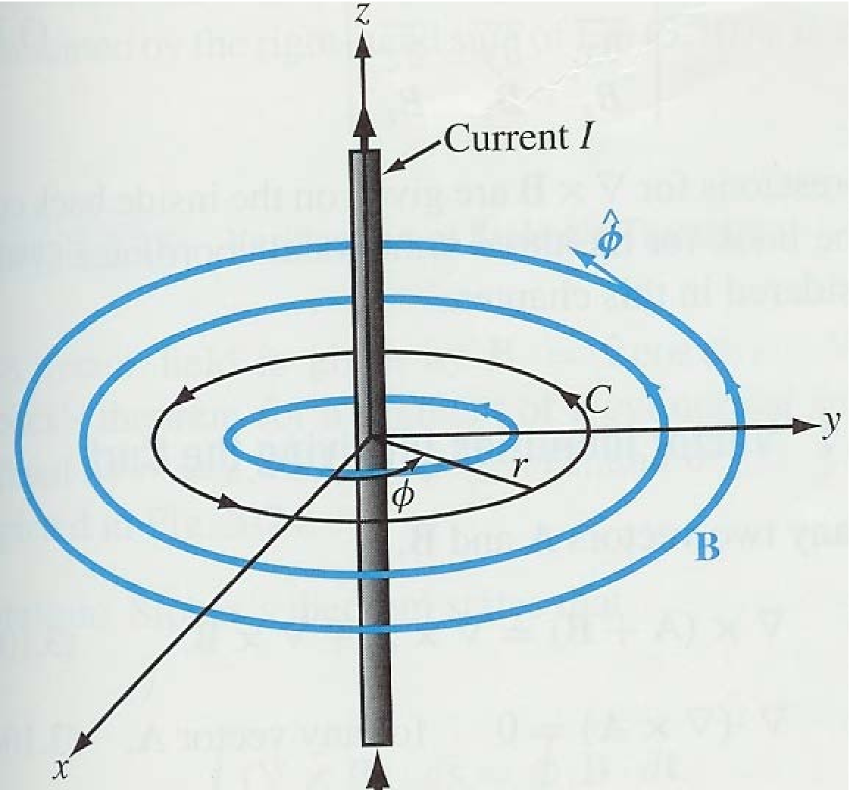
$$Circulation = \oint \vec{B} \bullet d\vec{l},$$

$$Circulation = \int_a^b \hat{x}B_0 \bullet \hat{x}dx + \int_b^c \hat{x}B_0 \bullet \hat{y}dy$$

$$+ \int_c^d \hat{x}B_0 \bullet \hat{x}dx + \int_d^a \hat{x}B_0 \bullet \hat{y}dy$$

$$= 0,$$

Circulation of a Vector

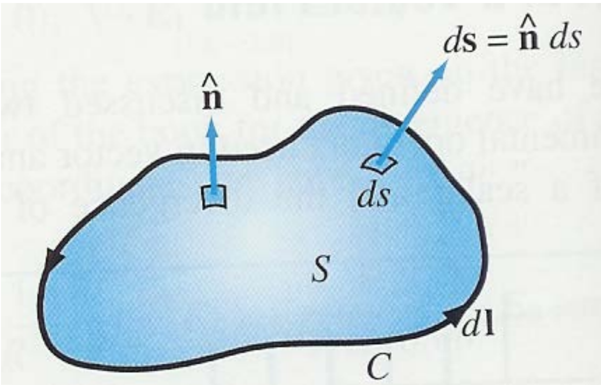


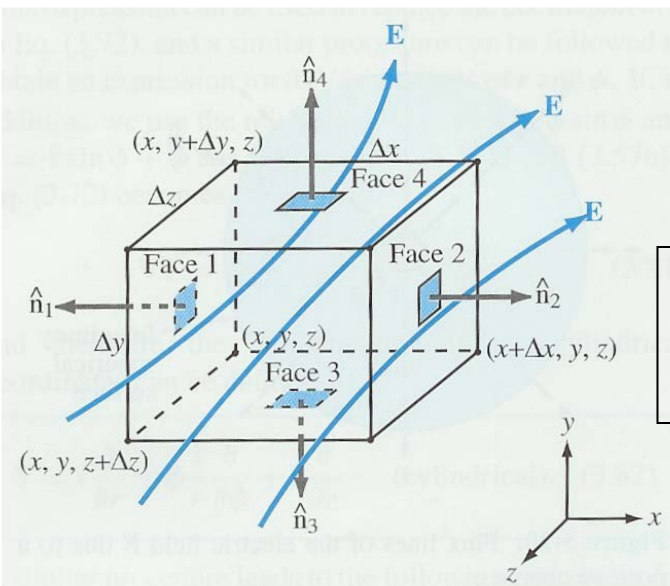
$$\vec{B} = \hat{\phi}\frac{\mu_0 I}{2\pi r},$$

$$Circulation = \oint \vec{B} \bullet d\vec{l} = \int_0^{2\pi} \hat{\phi}\frac{\mu_0 I}{2\pi r} \bullet \hat{\phi} d\phi,$$

$$= \mu_0 I,$$

Curl in Cartesian Coordinates





$$\int_{face2} \vec{n}_2 \vec{B} \bullet d\vec{l}_{y,z},$$

$$= \int_{face2} \hat{x}(B_{y+\Delta y}\Delta z - B_y \Delta z - B_{z+\Delta z}\Delta y + B_z \Delta y)$$

$$= \int \hat{x}(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z})\Delta y \Delta z,$$

$$\oint_c \vec{n}\vec{B} \bullet d\vec{l} = \int \hat{x}(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z})\Delta y \Delta z + \Delta x \Delta z + \int \hat{z}(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial z})\Delta y \Delta x$$

$$= \int [\hat{x}(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}) + \hat{y}(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}) + \hat{z}(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial z})] \bullet d\vec{s}$$

$$\int \hat{n}(\nabla \times B)\Delta s$$

$$\nabla \times \vec{B} \equiv \hat{x}(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}) + \hat{y}(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}) + \hat{z}(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial z})$$

Vector identities involving the curl

$$\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$$

$$\nabla \bullet (\nabla \times \vec{A}) = 0$$

$$\nabla \bullet (\nabla V) = 0$$

Stokes's theorem

$$\oint_c \vec{n}\vec{B} \bullet d\vec{l} = \int (\nabla \times B) \bullet d\vec{s}$$

Curls in Rectangular, Cylindrical and Spherical Coordinates

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

$$\nabla \times \vec{B} = \frac{1}{r} \begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}$$

$$\nabla \times \vec{B} = \frac{1}{R^2 \sin\theta} \begin{vmatrix} \hat{R} & R\hat{\theta} & R\sin\theta\hat{\phi} \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_R & RA_\theta & R\sin\theta A_\phi \end{vmatrix}$$

Laplacian Operator of a scalar

$$\nabla V = \hat{x}\frac{\partial V}{\partial x} + \hat{y}\frac{\partial V}{\partial y} + \hat{z}\frac{\partial V}{\partial z} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$$

$$\nabla \bullet (\nabla V) = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z$$

$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\nabla^2 V \equiv \nabla \bullet (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Laplacian Operator of a vector

$$\vec{E} = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z$$

$$\nabla^2 \vec{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\vec{E} = \hat{x}\frac{\partial^2 E_x}{\partial x^2} + \hat{y}\frac{\partial^2 E_y}{\partial y^2} + \hat{z}\frac{\partial^2 E_z}{\partial z^2}$$

$$\nabla^2 \vec{E} = \nabla(\nabla \bullet \vec{E}) - \nabla \times (\nabla \times \vec{E})$$

Maxwell equations

$$\nabla \bullet \vec{D} = \rho_v, \qquad \nabla \bullet \vec{B} = 0,$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \qquad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t},$$

E: electric field intensity

D: electric flux intensity

H: magnetic field intensity

B: magnetic flux intensity

$\vec{D} = \varepsilon \vec{E}, \quad \vec{B} = \mu \vec{H},$ ε: electrical permittivity; μ:magnetic permativity

ρ_v: electric charge density per unit volume; J: current density per unit area.

Electrostatics

$$\nabla \bullet \vec{D} = \rho_v,$$

$$\nabla \times \vec{E} = 0,$$

Magnetostatics

$$\nabla \bullet \vec{B} = 0,$$

$$\nabla \times \vec{H} = \vec{J},$$

Electrostatics

$$\nabla \bullet \vec{D} = \rho_v, \quad \nabla \times \vec{E} = 0,$$

Volume charge density

$$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v} = \frac{dq}{dv}, \qquad Q = \int_v \rho_v dv,$$

Surface charge density

$$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds},$$

Line charge density

$$\rho_l = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl},$$

Current density J

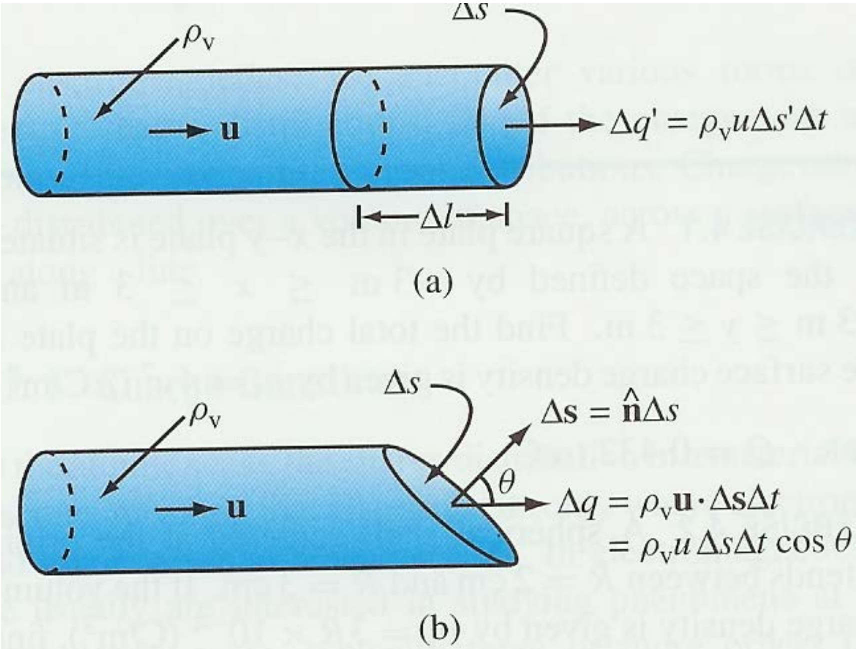


Figure 4-2: Charges with velocity **u** moving through a cross section $\Delta s'$ in (a) and Δs in (b).

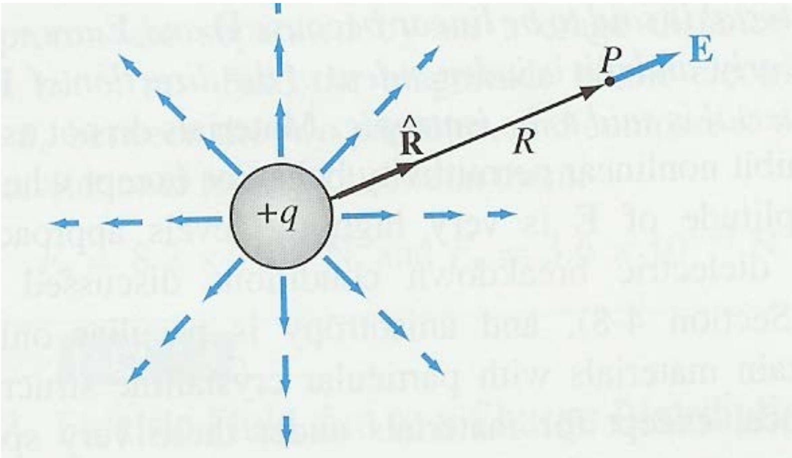
$$\Delta q = \rho_v \vec{u} \bullet \Delta \vec{s} \Delta t,$$

$$\Delta I = \frac{\Delta q}{\Delta t} = \rho_v \vec{u} \bullet \Delta \vec{s},$$

$$\boxed{\vec{J} = \rho_v \vec{u},}$$

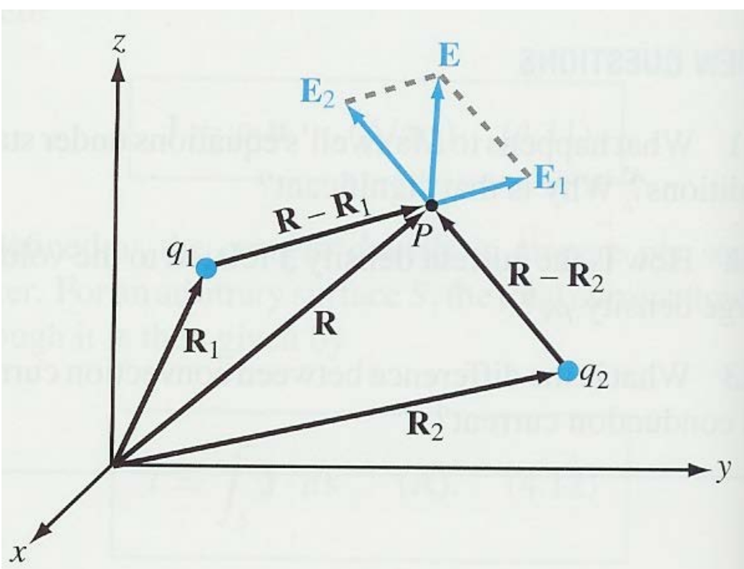
$$\boxed{I = \int_s \vec{J} \bullet d\vec{s},}$$

Coulomb's law



$$\vec{E} = \vec{R}\frac{q}{4\pi\varepsilon R^2},$$

$$\vec{F} = q'\vec{E},$$

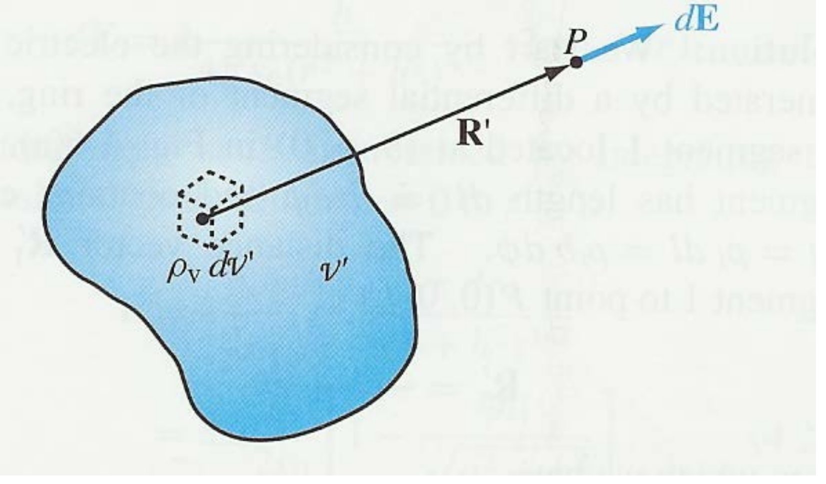


$$\vec{E}_1 = \frac{q_1(\vec{R}-\vec{R}_1)}{4\pi\varepsilon\left|\vec{R}-\vec{R}_1\right|^3}, \quad \vec{E}_2 = \frac{q_2(\vec{R}-\vec{R}_2)}{4\pi\varepsilon\left|\vec{R}-\vec{R}_2\right|^3},$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{1}{4\pi\varepsilon}\left[\frac{q_1(\vec{R}-\vec{R}_1)}{\left|\vec{R}-\vec{R}_1\right|^3} + \frac{q_2(\vec{R}-\vec{R}_2)}{\left|\vec{R}-\vec{R}_2\right|^3}\right],$$

$$\vec{E} = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{q_1(\vec{R}-\vec{R}_i)}{\left|\vec{R}-\vec{R}_i\right|^3},$$

Electric field due to a charge distribution



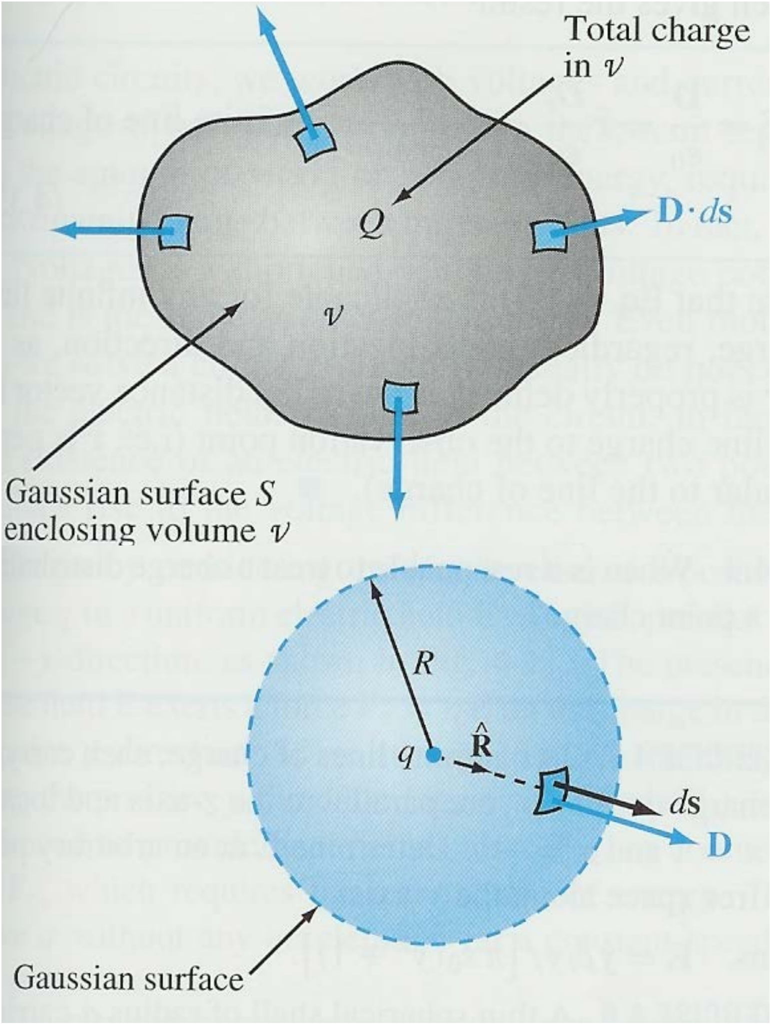
$$d\vec{E} = \hat{R}'\frac{dq}{4\pi\varepsilon R'^2},$$

$$\vec{E} = \int d\vec{E} = \frac{1}{4\pi\varepsilon}\int_{v'}\hat{R}'\frac{\rho_v dv}{R'^2},$$

$$\vec{E} = \int d\vec{E} = \frac{1}{4\pi\varepsilon}\int_{s}\hat{R}'\frac{\rho_s ds'}{R'^2},$$

$$\vec{E} = \int d\vec{E} = \frac{1}{4\pi\varepsilon}\int_{l}\hat{R}'\frac{\rho_l dl'}{R'^2},$$

Gauss's law



$$\nabla \bullet \vec{D} = \rho_v, \quad \int_v \nabla \bullet \vec{D} dv = \int_v \rho_v dv = Q,$$

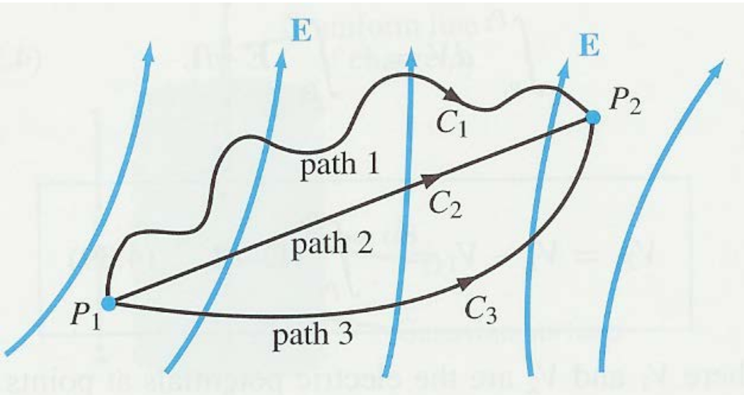
$$\int_v \nabla \bullet \vec{D} dv = \oint_s \vec{D} \bullet d\vec{s},$$

$$\boxed{\oint_s \vec{D} \bullet d\vec{s} = Q,}$$

Gauss's law

16.360 Lecture 19

Electrical scalar potential



$$dW = -\vec{F}_e \bullet d\vec{l} = -q\vec{E} \bullet d\vec{l},$$

$$dW = q dV,$$

$$dV = -\vec{E} \bullet d\vec{l},$$

$$V_{21} = V_2 - V_1 = \int_{P_1}^{P_2} dV = \int_{P_1}^{P_2} -\vec{E} \bullet d\vec{l},$$

$$0 = \oint_C dV = -\oint_C \vec{E} \bullet d\vec{l} = 0,$$

$$\oint_s (\nabla \times \vec{E}) \bullet d\vec{s} = \oint_C \vec{E} \bullet d\vec{l} = 0,$$

Electrical potential due to point charge

$$\vec{E} = \vec{R}\frac{q}{4\pi\varepsilon R^2}, \qquad V = \int_{\infty}^{R} -\vec{E} \bullet d\vec{l}$$

$$V = \int_{\infty}^{R} -\vec{E} \bullet d\vec{l} = -\int_{\infty}^{R} \hat{R}\frac{q}{4\pi\varepsilon R^2} \bullet \hat{R}dR = \frac{q}{4\pi\varepsilon R},$$

$$V(\vec{R}) = \frac{q}{4\pi\varepsilon\left|\vec{R}-\vec{R}_1\right|},$$

Electrical potential due to continuous distributions

$$V(\vec{R}) = \frac{1}{4\pi\varepsilon}\int_{v'}\frac{\rho_{v'}}{\left|\vec{R}-\vec{R}'\right|}dv', \quad V(\vec{R}) = \frac{1}{4\pi\varepsilon}\int_{s'}\frac{\rho_{s'}}{\left|\vec{R}-\vec{R}'\right|}ds',$$

$$V(\vec{R}) = \frac{1}{4\pi\varepsilon}\int_{l'}\frac{\rho_{l'}}{\left|\vec{R}-\vec{R}'\right|}dl',$$

Electric field as a function of Electrical potential

$$dV = -\vec{E} \bullet d\vec{l}, \quad dV = \nabla V \bullet d\vec{l},$$

$$\vec{E} = -\nabla V,$$

Poison's equation

$$\nabla \bullet \vec{D} = \rho_v, \quad \nabla \bullet \vec{E} = \frac{\rho_v}{\varepsilon},$$

$$\nabla \bullet \nabla V = -\frac{\rho_v}{\varepsilon}, \qquad \boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon},}$$ Poison's equation

$$\boxed{\nabla^2 V = 0,}$$ Laplace's equation

Electrical properties of material

- conductor
- dielectric
- semiconductor

Material	Conductivity, σ (S/m)
Conductors	
Silver	6.2×10^7
Copper	5.8×10^7
Gold	4.1×10^7
Aluminum	3.5×10^7
Iron	10^7
Mercury	10^6
Carbon	3×10^4
Semiconductors	
Pure germanium	2.2
Pure silicon	4.4×10^{-4}
Insulators	
Glass	10^{-12}
Paraffin	10^{-15}
Mica	10^{-15}
Fused quartz	10^{-17}

Conductors

Electron drift velocity $u_e = -\mu_e \vec{E}$

Hole drift velocity $u_h = \mu_h \vec{E}$

Conducting current

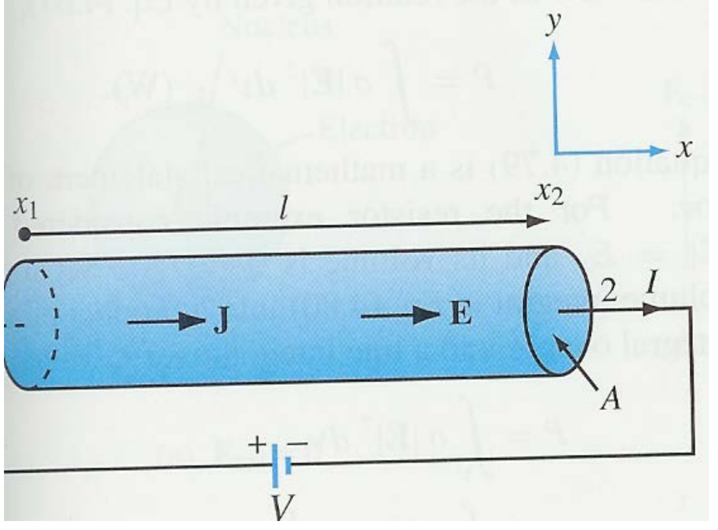
$$\vec{J} = \vec{J}_e + \vec{J}_h = \rho_{ve} u_e + \rho_{vh} u_h = (-\rho_{ve}\mu_e + \rho_{vh}\mu_h)\vec{E},$$

$$\sigma = -\rho_{ve}\mu_e + \rho_{vh}\mu_h,$$

Point form of Ohm's law

$$\boxed{\vec{J} = \sigma\vec{E},}$$

Resistance



$$V = V_1 - V_2 = -\int_{x_2}^{x_1} \vec{E} \bullet d\vec{l} = E_x l,$$

$$I = \int_A \vec{J} \bullet d\vec{s} = \int_A \sigma\vec{E} \bullet d\vec{s} = \sigma E_x A,$$

$$R = \frac{V}{I} = \frac{l}{\sigma A},$$

General form

$$R = \frac{V}{I} = \frac{-\int_{x_2}^{x_1} \vec{E} \bullet d\vec{l}}{\int_A \vec{J} \bullet d\vec{s}} = \frac{-\int_{x_2}^{x_1} \vec{E} \bullet d\vec{l}}{\int_A \sigma\vec{E} \bullet d\vec{s}},$$

Joule's law

$$\Delta W = F_e \Delta l_e + F_h \Delta l_h,$$

$$\Delta P = \frac{\Delta W}{\Delta t} = F_e \frac{\Delta l_e}{\Delta t} + F_h \frac{\Delta l_h}{\Delta t}$$

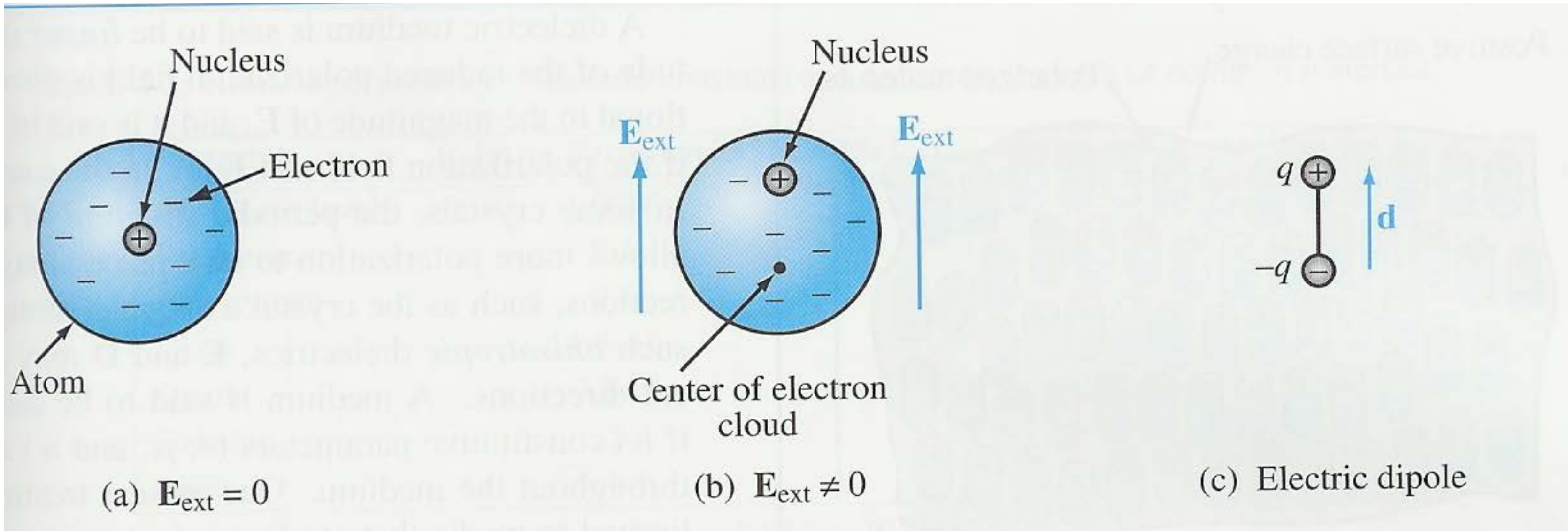
$$= F_e u_e + F_h u_h = (\rho_{ve} u_e \vec{E} + \rho_{vh} u_h \vec{E}) \Delta v$$

$$= \vec{J} \bullet \vec{E} \Delta v,$$

General form

$$P = \int_v \vec{J} \bullet \vec{E} dv,$$

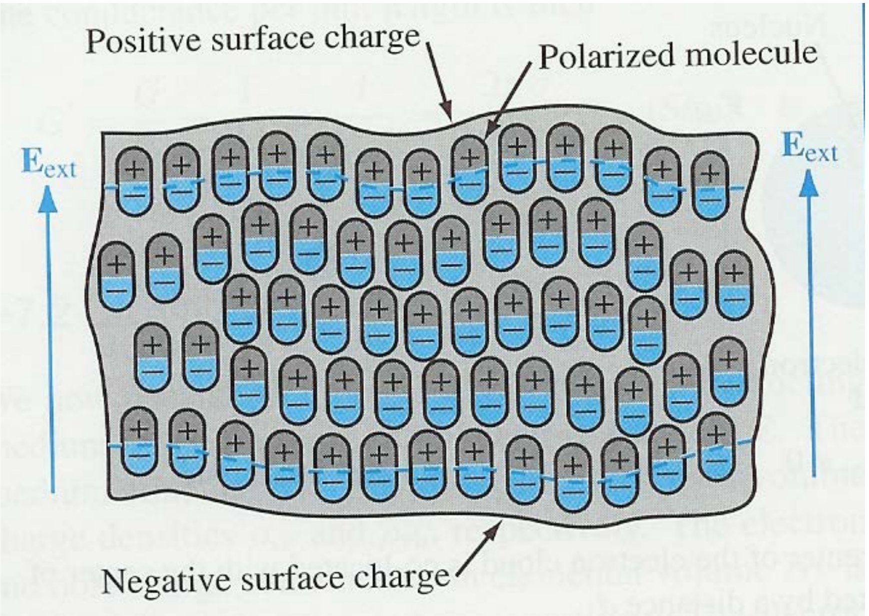
Dielectrics



Electrical field induced polarization

16.360 Lecture 20

Dielectrics



$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P},$$

P: electric polarization field

For homogeneous material:

$$\vec{P} = \varepsilon_0 \chi_e \vec{E},$$

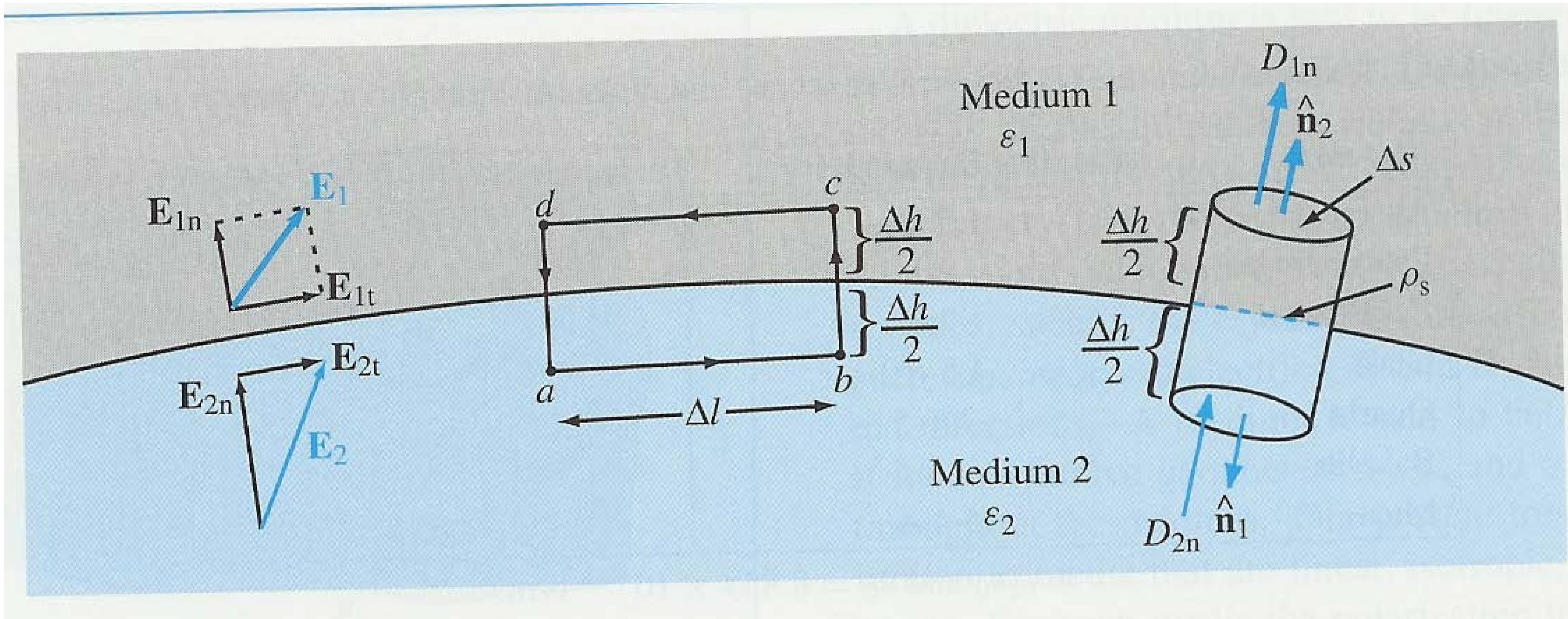
$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} = \varepsilon \vec{E},$$

$\varepsilon = \varepsilon_0 (1 + \chi_e),$ Electric susceptibility

Relative permittivity: $\varepsilon_r = \dfrac{\varepsilon}{\varepsilon_0} = (1 + \chi_e),$

Dielectric breakdown

Electric boundary condition



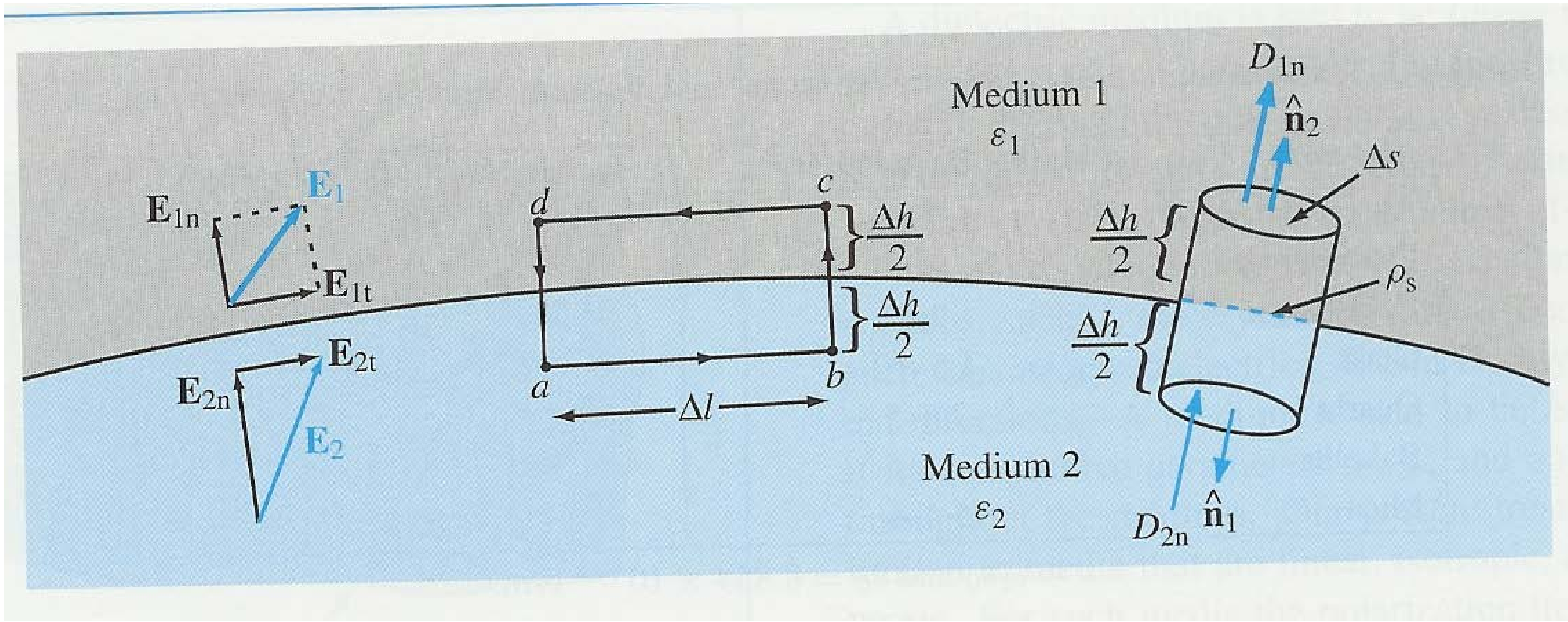
$$\oint_C \vec{E} \bullet d\vec{l} = \lim_{\Delta h \to 0}[\int_a^b \vec{E}_2 \bullet d\vec{l} + \int_c^d \vec{E}_1 \bullet d\vec{l}] = 0;$$

the tangential component is continuous across the boundary of two media.

$$\vec{E}_1 = \vec{E}_{1t} + \vec{E}_{1n}, \qquad \vec{E}_{1t}\Delta l - \vec{E}_{2t}\Delta l = 0,$$

$$\vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n}, \qquad \vec{E}_{1t} = \vec{E}_{2t},$$

Electric boundary condition



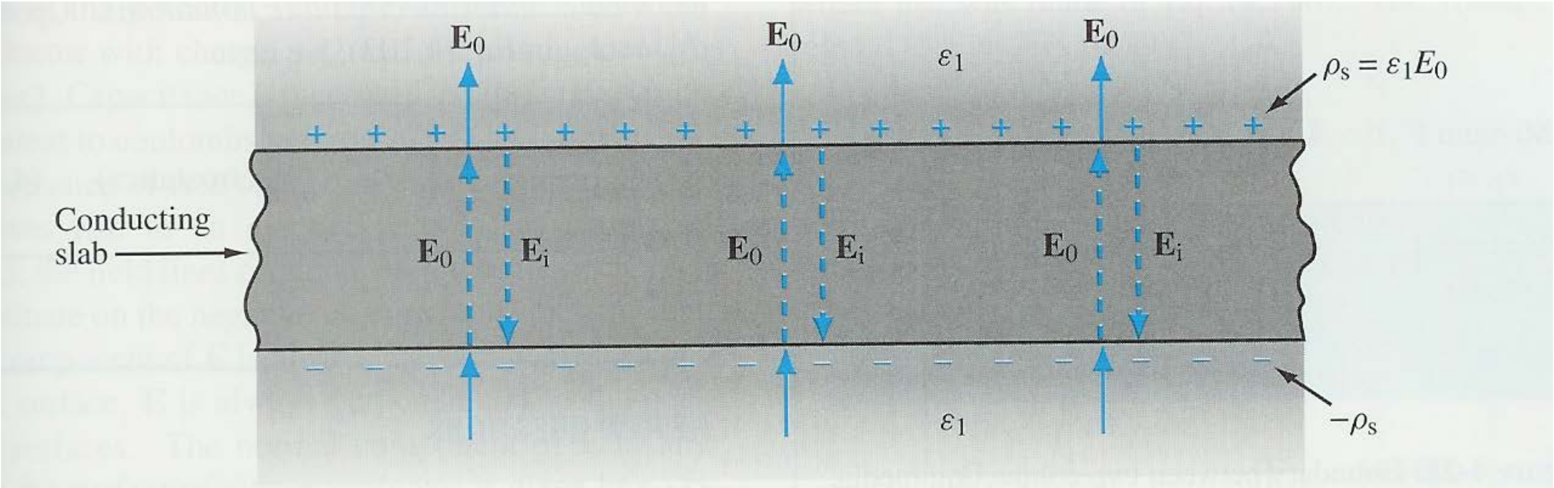
$$\oint_C \vec{D} \bullet d\vec{s} = \lim_{\Delta h \to 0} [\int_{top} \vec{D} \bullet d\vec{s} + \int_{bottom} \vec{D} \bullet d\vec{s}] = \rho_s \Delta s;$$

the normal component of D changes, the amount of change is equal to the surface Charge density.

$$\vec{D}_{1n}\Delta s - \vec{D}_{2n}\Delta s = \rho_s \Delta s,$$

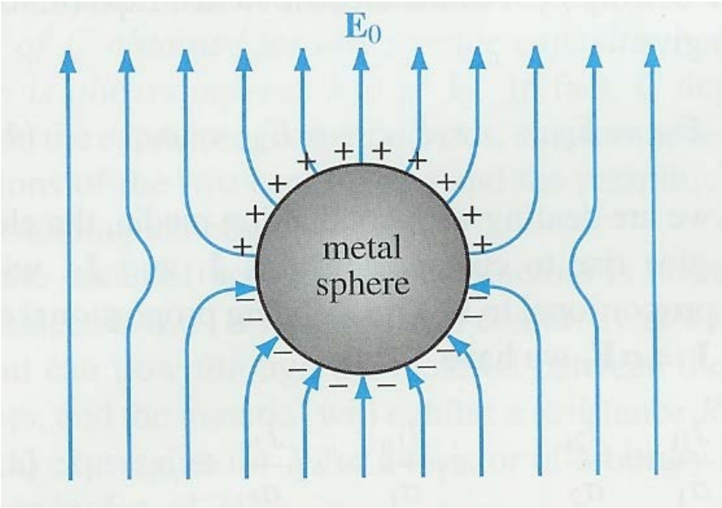
$$\vec{D}_{1n} - \vec{D}_{2n} = \rho_s,$$

Dielectric-Conductor boundary

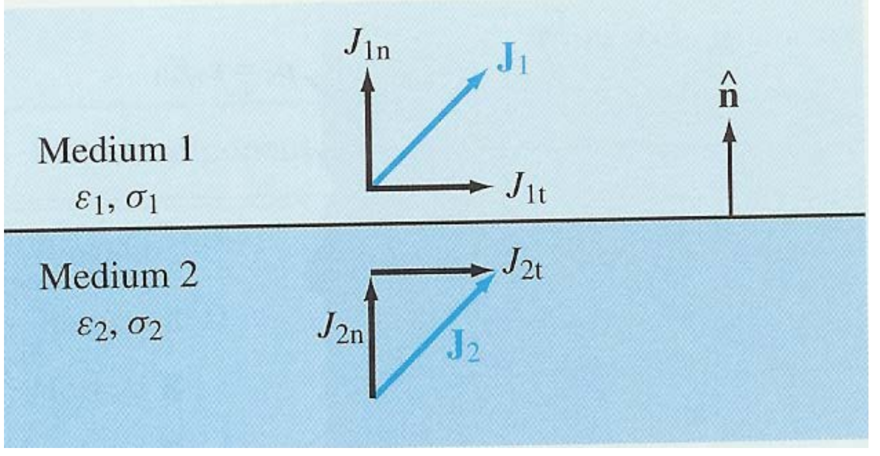


$$\vec{E}_{1t} = \vec{E}_{2t} = 0,$$

$$\vec{D}_{1n} = \rho_s,$$



Conductor-Conductor boundary

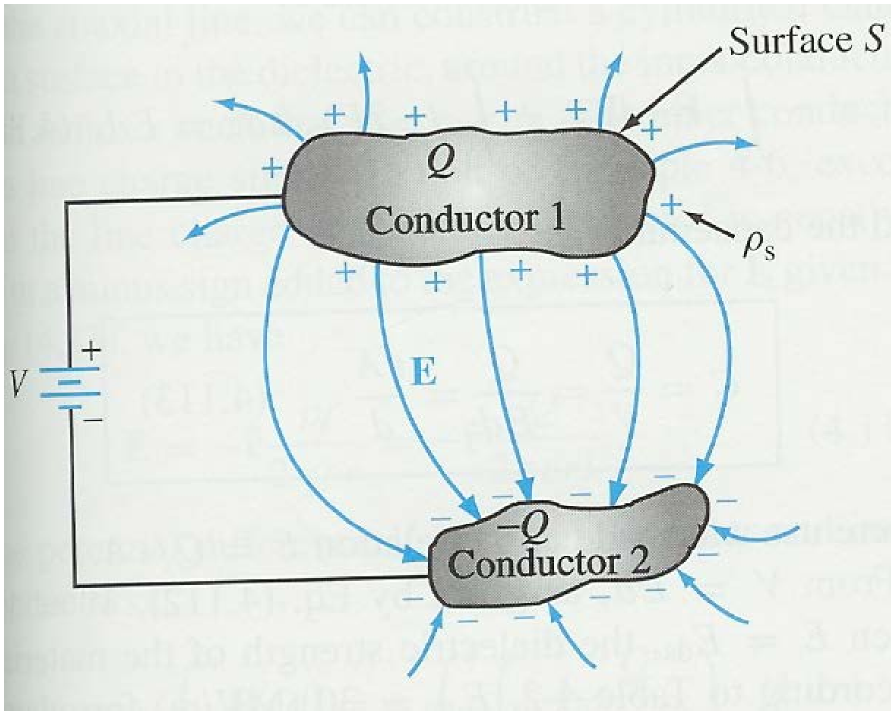


$$\vec{E}_{1t} = \vec{E}_{2t}, \qquad \vec{D}_{1n} - \vec{D}_{2n} = \varepsilon_1 \vec{E}_{1n} - \varepsilon_2 \vec{E}_{2n} = \rho_s,$$

$$\frac{\vec{J}_{1t}}{\sigma_1} = \frac{\vec{J}_{2t}}{\sigma_2}, \qquad \varepsilon_1 \frac{\vec{J}_{1n}}{\sigma_1} - \varepsilon_2 \frac{\vec{J}_{2n}}{\sigma_2} = \rho_s,$$

$$\vec{J}_{1n}\left(\frac{\varepsilon_1}{\sigma_1} - \frac{\varepsilon_2}{\sigma_2}\right) = \rho_s,$$

Capacitance



$$C = \frac{Q}{V},$$

$$Q = \int_s \varepsilon \vec{E} \bullet d\vec{s},$$

$$V = -\int_l \vec{E} \bullet d\vec{l}$$

$$C = \frac{\int_s \varepsilon \vec{E} \bullet d\vec{s}}{-\int_l \vec{E} \bullet d\vec{l}},$$

$$R = \frac{V}{I} = \frac{-\int_{x_2}^{x_1} \vec{E} \bullet d\vec{l}}{\int_A \vec{J} \bullet d\vec{s}} = \frac{-\int_{x_2}^{x_1} \vec{E} \bullet d\vec{l}}{\int_A \sigma \vec{E} \bullet d\vec{s}},$$

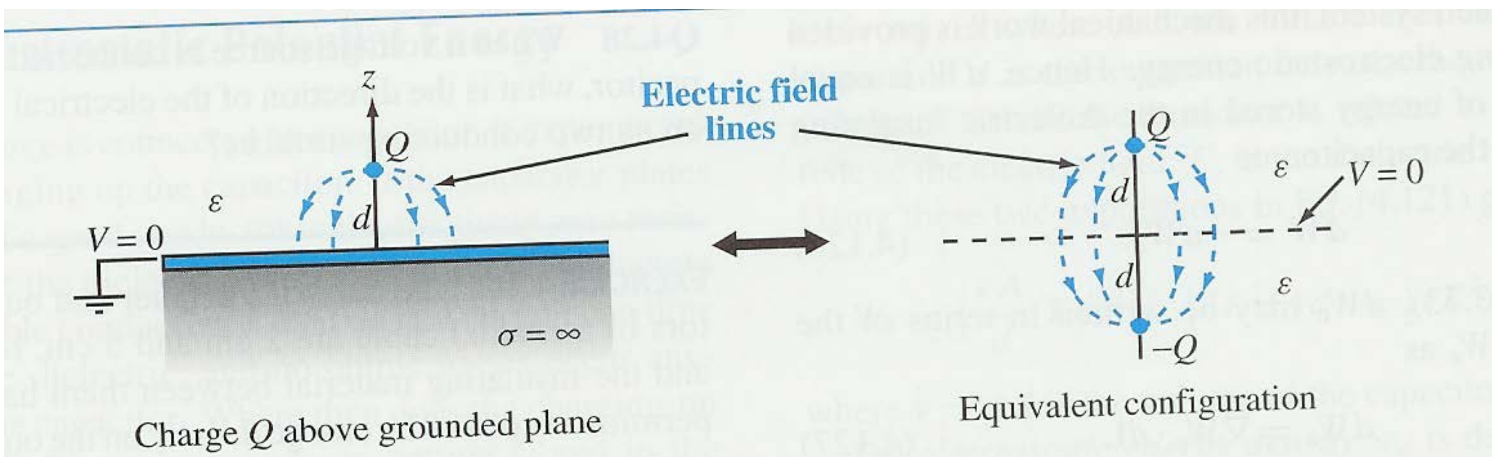
$$RC = \frac{\varepsilon}{\sigma},$$

Electrostatic Potential Energy

$$W_e = \frac{1}{2}\vec{D} \bullet \vec{E}, \quad dW_e = -\vec{F} \bullet d\vec{l} = \nabla W_e \bullet d\vec{l},$$

$$\vec{F} = -\nabla W_e,$$

Image Method



Any given charge above an infinite, perfect conducting plane is electrically equivalent to the combination of the give charge and it’s image with conducting plane removed.

Magnetostatics

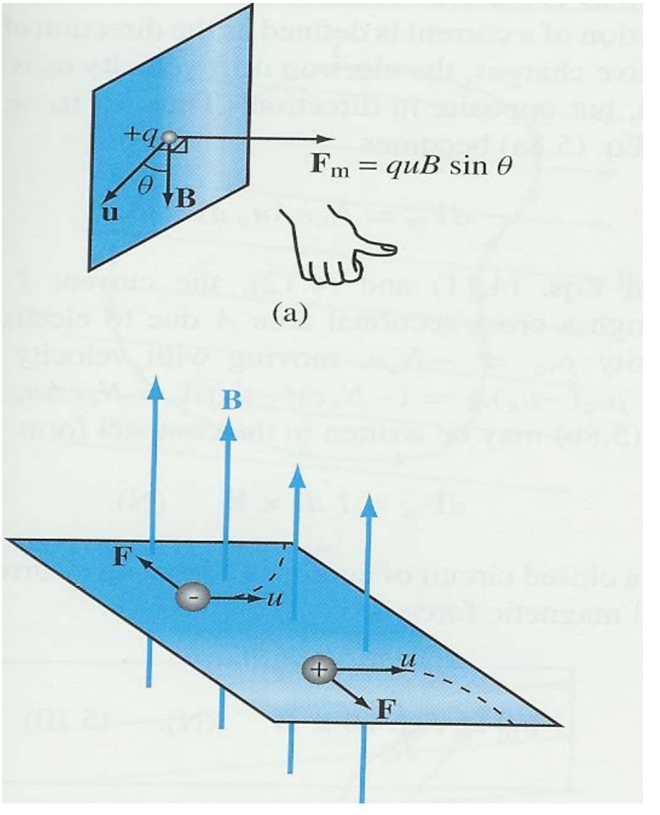
$$\nabla \bullet \vec{B} = 0, \qquad \vec{B} = \mu \vec{H},$$

$$\nabla \times \vec{H} = \vec{J}, \qquad \mu = \mu_r \mu_0,$$

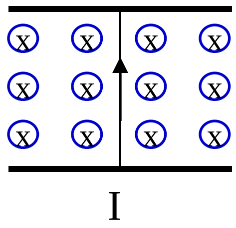
Magnetic forces and Torques

$$F_m = q\vec{u} \times \vec{B},$$

$$F = F_e + F_m = q\vec{E} + q\vec{u} \times \vec{B},$$

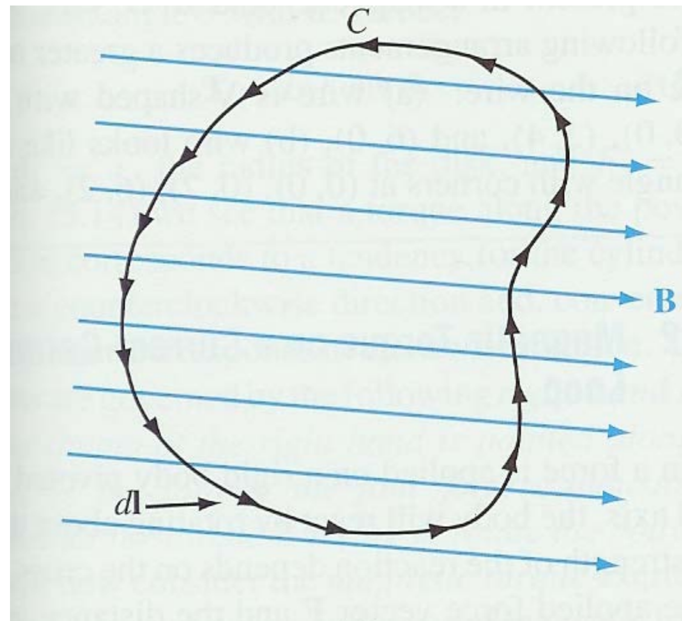


Magnetic forces on a current-carrying conductor

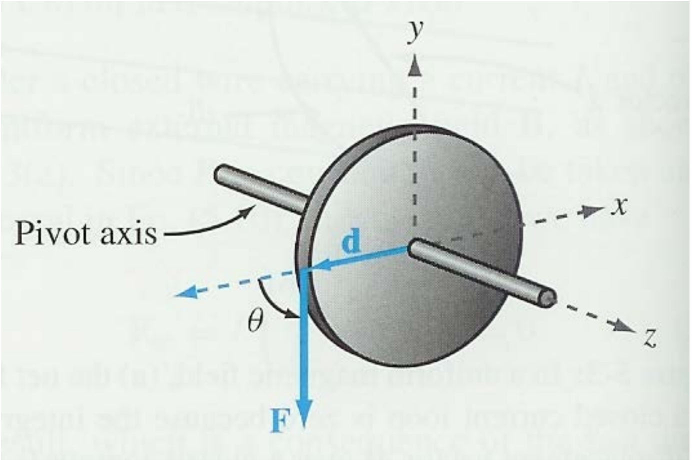


$$dF_m = I d\vec{l} \times \vec{B},$$

$$F_m = I\left(\oint_C d\vec{l} \times \vec{B}\right),$$



Magnetic torques on a current-carrying conductor



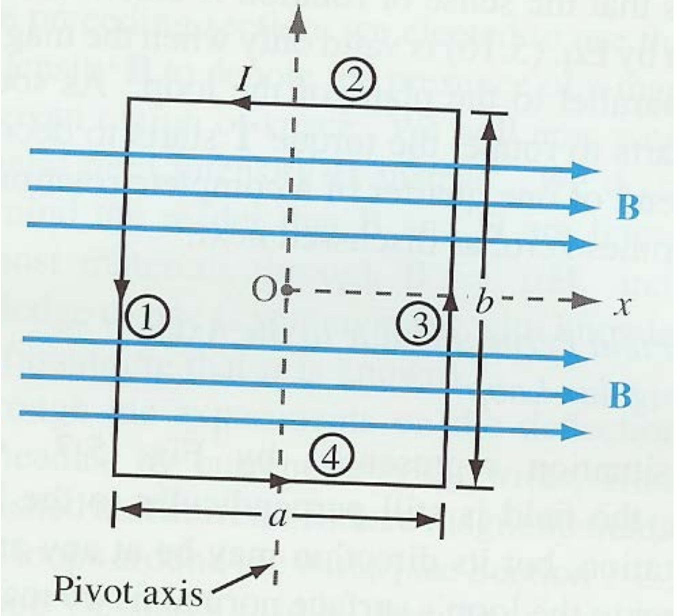
$$\vec{T} = \vec{d} \times \vec{F},$$

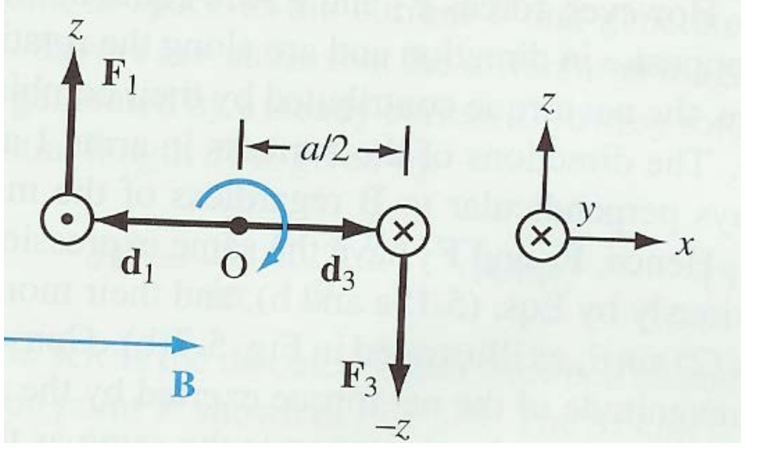
$$\vec{F}_1 = I(-\hat{y}b) \times (\hat{x}B_0) = \hat{z}IbB_0,$$

$$\vec{F}_3 = I(\hat{y}b) \times (\hat{x}B_0) = -\hat{z}IbB_0,$$

$$\vec{T} = \vec{d}_1 \times \vec{F}_1 + \vec{d}_3 \times \vec{F}_3 = (-\hat{x}\frac{a}{2}) \times (\hat{z}IbB_0)$$

$$+ (\hat{x}\frac{a}{2}) \times (-\hat{z}IbB_0) = \hat{y}IabB_0 = \hat{y}IAB_0,$$

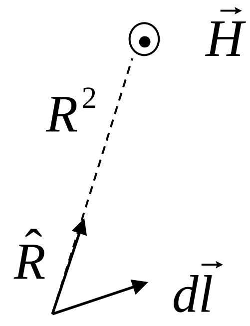




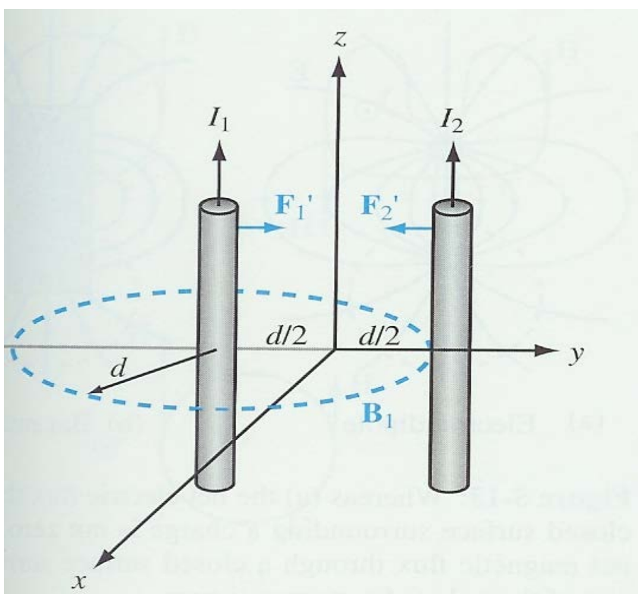
The Biot-Savart Law

$$d\vec{H} = \frac{I}{4\pi}\frac{d\vec{l} \times \hat{R}}{R^2},$$

$$\vec{H} = \frac{I}{4\pi}\oint_l \frac{d\vec{l} \times \hat{R}}{R^2},$$



Magnetic force between two parallel conductor



$$\vec{B}_1 = -\hat{x}\frac{\mu_0 I_1}{2\pi d},$$

$$\vec{F}_2 = I_2 l\hat{z} \times \vec{B}_1 = I_2 l\hat{z} \times (-\hat{x})\frac{\mu_0 I_1}{2\pi d}$$

$$= -\hat{y}\frac{\mu_0 I_1 I_2 l}{2\pi d},$$

Gauss's Law for Magnetism

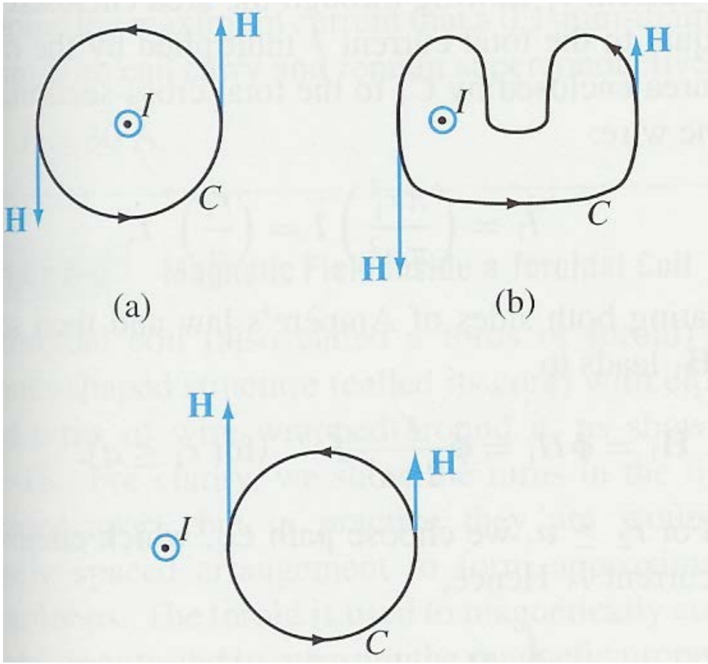
$$\nabla \bullet \vec{D} = \rho_v, \quad \oint_S \vec{D} \bullet d\vec{s} = Q_v,$$

$$\nabla \bullet \vec{B} = 0, \quad \oint_S \vec{B} \bullet d\vec{s} = Q_v,$$

Ampere's Law

$$\nabla \times \vec{H} = \vec{J}, \quad \int_S (\nabla \times \vec{H}) \cdot d\vec{s} = \int_S \vec{J} \cdot d\vec{s} = I,$$

$$\oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{s} = I,$$



16.360 Lecture 22

Vector Magnet Potential

$$\nabla \bullet \vec{B} = 0, \qquad \vec{B} = \mu\vec{H},$$

$$\nabla \times \vec{H} = \vec{J}, \qquad \mu = \mu_r \mu_0,$$

For any vector A

$\nabla \cdot (\nabla \times \vec{A}) = 0,$ $\quad \nabla \bullet \vec{B} = 0,$ Define: $\vec{B} \equiv \nabla \times \vec{A},$ with $\nabla \cdot \vec{A} = 0,$

$$\nabla \times \vec{H} = \vec{J}, \qquad \nabla \times (\nabla \times \vec{A}) = \mu\vec{J},$$

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = -\nabla^2 \vec{A} = \mu\vec{J},$$

$$\nabla^2 \vec{A} = -\mu\vec{J},$$

Vector Poison's equation $\quad \vec{A} = \frac{\mu}{4\pi}\int_{v'} \frac{\vec{J}}{R'} dv',$

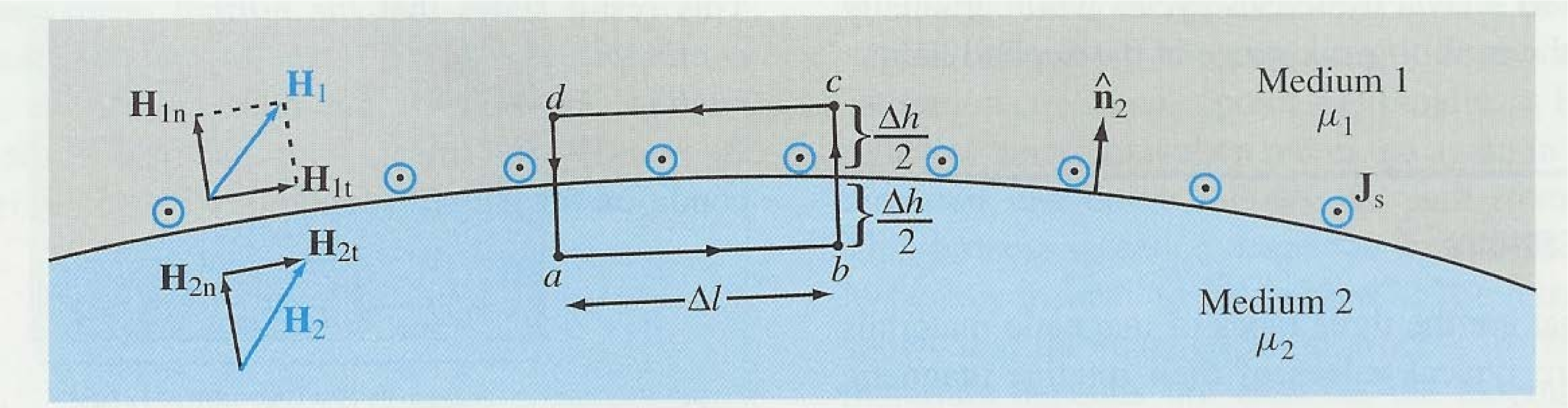
Magnetic boundary conditions

$$\oint_C \vec{D} \bullet d\vec{s} = \rho_s; \quad \vec{D}_{1n} - \vec{D}_{2n} = \rho_s,$$

$$\nabla \cdot \vec{B} = 0, \quad \oint_C \vec{B} \bullet d\vec{s} = 0; \quad \vec{B}_{1n} - \vec{B}_{2n} = 0,$$

$$\oint_C \vec{E} \bullet d\vec{l} = 0; \quad \vec{E}_{1t} = \vec{E}_{2t},$$

$$\nabla \times \vec{H} = \vec{J}, \quad \oint_C \vec{H} \bullet d\vec{l} = J; \quad \vec{H}_{1t} - \vec{H}_{2t} = J,$$



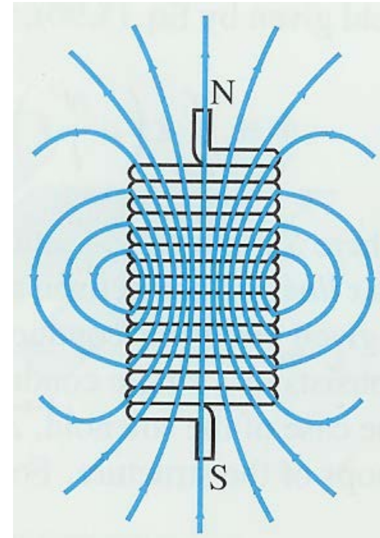
Inductance

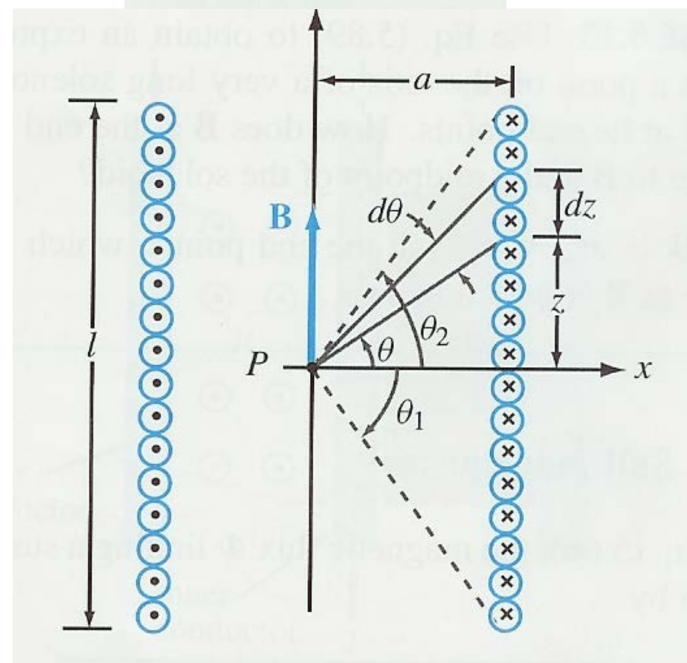
Magnetic filed in a Solenoid

$$\vec{H} = \hat{z}\frac{I' a^2}{2(a^2+z^2)^{3/2}},$$

$$d\vec{B} = \mu dH = \hat{z}\frac{\mu n I' a^2}{2(a^2+z^2)^{3/2}}dz,$$

$$\vec{B} = \hat{z}\frac{\mu n I'}{2}\sin(\theta_2 - \theta_1),$$





Self Inductance

$$\Phi = \int_s \vec{B} \cdot d\vec{s} = \mu \frac{NIS}{l},$$

$$\Lambda = N\Phi = N\int_s \vec{B} \cdot d\vec{s} = \mu \frac{N^2 IS}{l},$$

$$L = \frac{\Lambda}{I} = \frac{N\int_s \vec{B} \cdot d\vec{s}}{I} = \mu \frac{N^2 S}{l},$$

Mutual Inductance

$$L_{12} = \frac{\Lambda_{12}}{I_1} = \frac{N_2 \int_s \vec{B}_1 \cdot d\vec{s}}{I_1}, \qquad L_{21} = \frac{\Lambda_{21}}{I_2} = \frac{N_1 \int_s \vec{B}_2 \cdot d\vec{s}}{I_2},$$

Magnetic Energy

$$W_m = \frac{1}{2}\int_{v'} \vec{B} \cdot \vec{H} dv',$$

Static field

$$\nabla \bullet \vec{D} = \rho_v, \qquad \vec{E} = \int d\vec{E} = \frac{1}{4\pi\varepsilon}\int_{v'} \hat{R}' \frac{\rho_v dv}{R'^2},$$

$$\nabla \times \vec{E} = 0,$$

$$\nabla \bullet \vec{B} = 0, \qquad \vec{H} = \frac{I}{4\pi}\oint_l \frac{d\vec{l} \times \hat{R}}{R^2},$$

$$\nabla \times \vec{H} = \vec{J},$$

Dynamic Field

$$\nabla \bullet \vec{D} = \rho_v, \qquad \nabla \bullet \vec{B} = 0,$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \qquad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t},$$

Table 6-1: Maxwell's equations.

Reference	Differential Form	Integral Form	
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$	(6.1)
Faraday's law	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$	(6.2)*
No magnetic charges (Gauss's law for magnetism)	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$	(6.3)
Ampère's law	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{s}$	(6.4)

*For a stationary surface S.

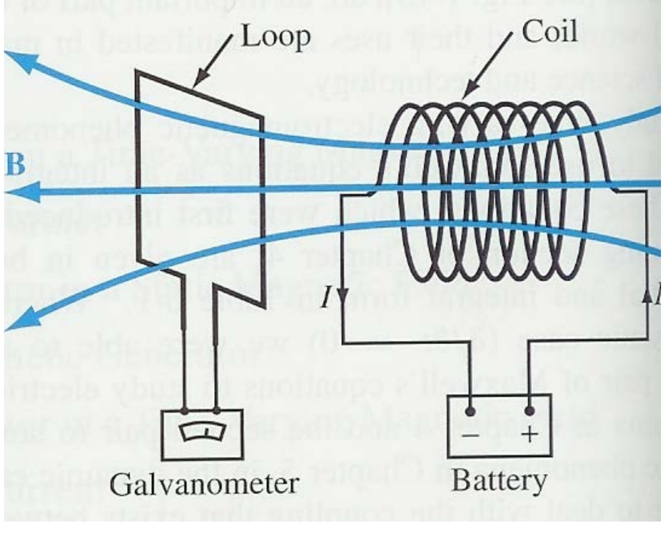
Faraday's Law

$$\oint_C \vec{E} \bullet d\vec{l} = -\oint_s \frac{\partial \vec{B}}{\partial t} \bullet d\vec{s} \qquad \Phi = \int_s \vec{B} \cdot d\vec{s},$$

$$= -\frac{\partial}{\partial t} \oint_s \vec{B} \bullet d\vec{s}, = -\frac{\partial \Phi}{\partial t},$$

Electromotive force

$$V_{emf} = \oint_C \vec{E} \bullet d\vec{l} = -\oint_s \frac{\partial \vec{B}}{\partial t} \bullet d\vec{s} = -\frac{\partial \Phi}{\partial t},$$



$$V_{emf} = V_{emf}^{tr} + V_{emf}^{m},$$

Stationary Loop in a Time-varying Magnetic field

$$V_{emf}^{tr} = -\oint_s \frac{\partial \vec{B}}{\partial t} \bullet d\vec{s}, \qquad I = \frac{V_{emf}^{tr}}{R_i + R} = -\oint_s \frac{\partial \vec{B}}{\partial t} \bullet d\vec{s},$$

Lenz's law

$$V_{emf} = \oint_C \vec{E} \bullet d\vec{l} = -\oint_s \frac{\partial \vec{B}}{\partial t} \bullet d\vec{s} = \oint_s (\nabla \times \vec{E}) \bullet d\vec{s},$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$ Faraday's law, differential form

An example:

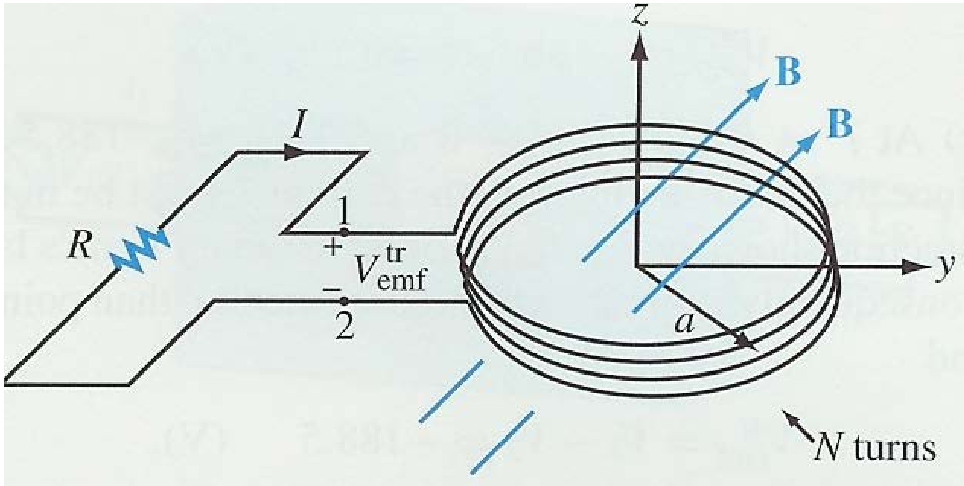
$$\vec{B} = B_0(\hat{y}2 + \hat{z}3)\sin\omega t,$$

(a) The magnetic flux link of a single turn of the inductor.

(b) The transformer emf,.

(c) The polarity of the emf.

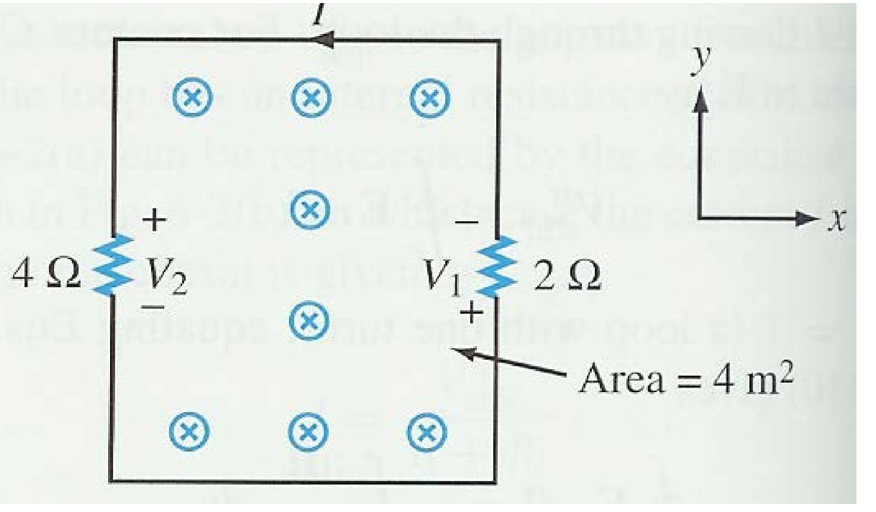
(d) The induced current.



Example II

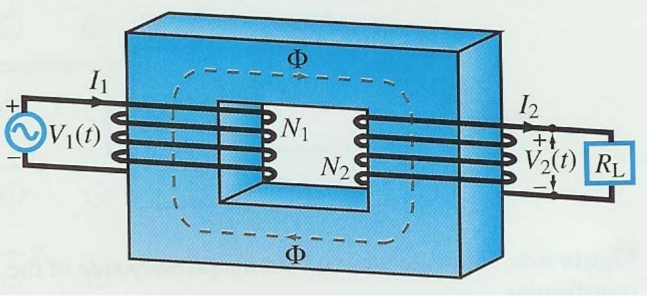
$$\vec{B} = -\hat{z}0.3t,$$

Determine the voltage drops across the two resistors



The ideal Transformer properties:

- $\mu = \infty$
- I = 0 in the core.
- The magnetic flux is confined within the core

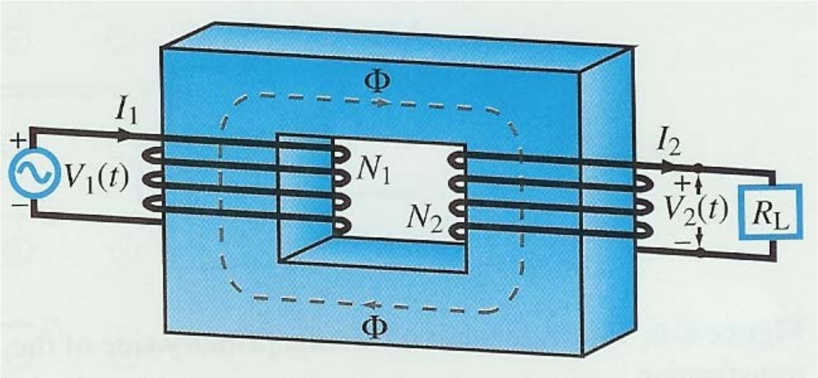


Questions:

- I = ?, with applied voltage of V1and with RL
- V2, and I2=?

Voltage transformer:

$$V_1 = -N_1 \frac{d\Phi}{dt}, \quad V_2 = -N_2 \frac{d\Phi}{dt}, \quad \frac{V_2}{V_1} = \frac{N_2}{N_1},$$

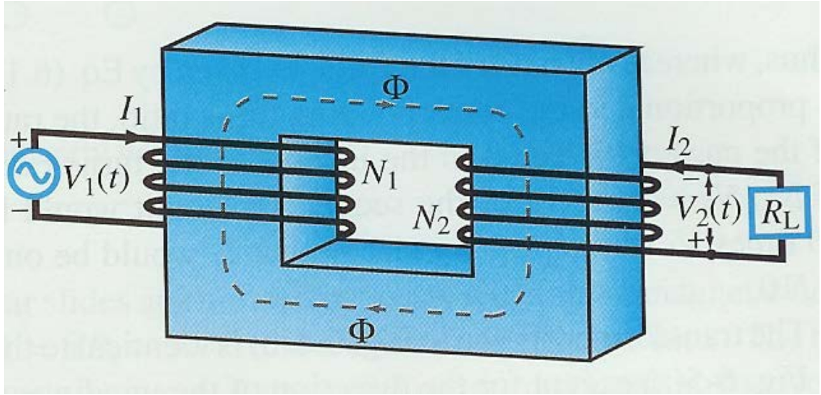


Power relations:

$P_1 = P_2,$ Why?

Current transformer:

$$P_1 = V_1 I_1, \quad P_2 = V_2 I_2, \quad \frac{I_2}{I_1} = \frac{N_1}{N_2},$$



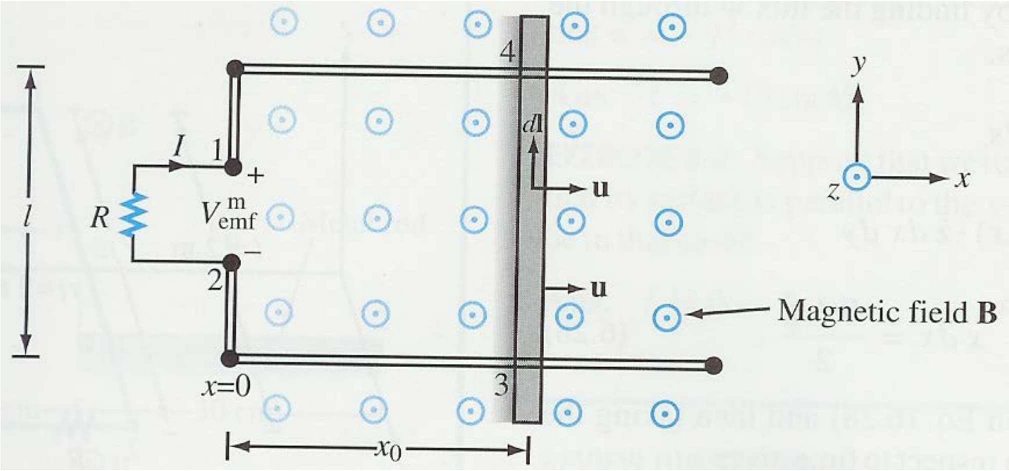
Impedance transformer:

$$R_1 = V_1 / I_1, \quad R_2 = V_2 / I_2, \quad \frac{R_1}{R_2} = (\frac{N_1}{N_2})^2,$$

$$R_{in} = (\frac{N_1}{N_2})^2 R_L,$$

Moving conductor in a static magnetic field:

$$V_{emf} = \oint_C \vec{E} \bullet d\vec{l} = -\oint_s \frac{\partial \vec{B}}{\partial t} \bullet d\vec{s} = -\frac{\partial \Phi}{\partial t},$$



$$V_{emf} = V_{emf}^{tr} + V_{emf}^{m},$$

$$V_{emf}^{m} = -\frac{\partial \Phi}{\partial t} = -\frac{\partial (\vec{B} \cdot d\vec{s})}{\partial t} = (\vec{u} \times \vec{B}) \cdot d\vec{l},$$

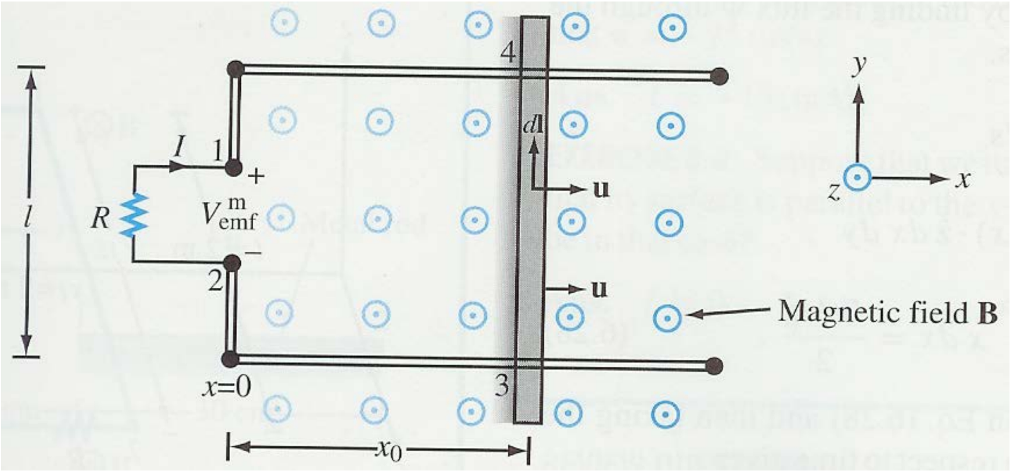
$$d\vec{s} = \vec{w} \times d\vec{l},$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{A} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}),$$

Another way to look at it:

$$F_m = q(\vec{u} \times \vec{B}), \quad E_m = \frac{F_m}{q},$$

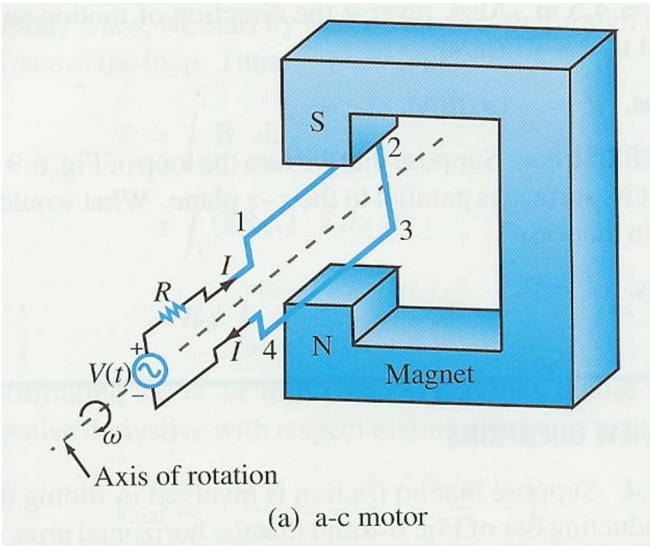
$$V_{emf}^m = \int_{l_1}^{l_2} E_m \cdot d\vec{l} = \int_{l_1}^{l_2} (\vec{u} \times \vec{B}) \cdot d\vec{l},$$

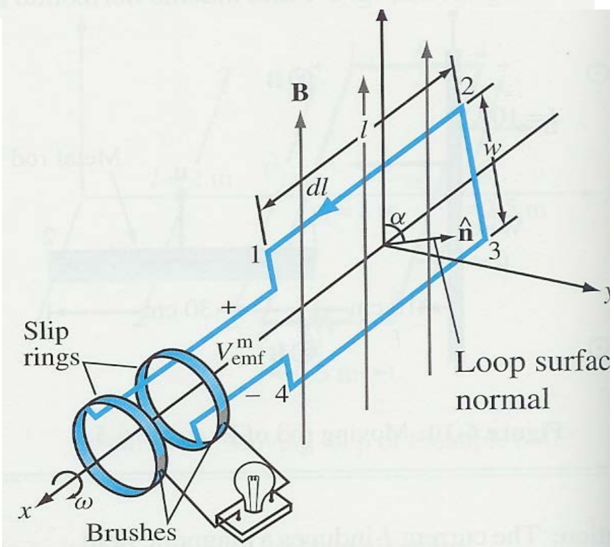


Next lecture:

- The electromagnetic generator
- Moving conductor in a time varying magnetic field

The electromagnetic generator





$$\Phi = \int_S \vec{B} \cdot d\vec{s} = B_0 A \cos(\alpha)$$

$$= B_0 A \cos(\omega t + C_0),$$

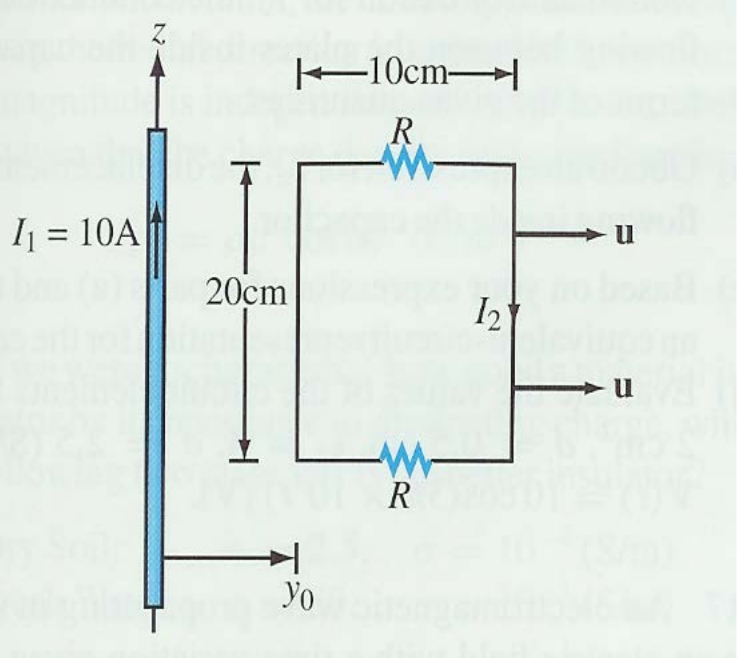
$$V_{emf} = -\frac{d\Phi}{dt} = -\frac{d}{dt} B_0 A \cos(\omega t + C_0)$$

$$= A\omega B_0 \sin(\omega t + C_0),$$

Moving conductor in a time-varying magnetic field

$$\oint_C \vec{E}\cdot d\vec{l} = V_{emf} = V_{emf}^{tr} + V_{emf}^{m} = -\int_s \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} + \oint_C (\vec{u}\times\vec{B})\cdot d\vec{l},$$

Example:



$$I_1 = 10A, \quad \vec{u} = 5\hat{y}, \quad R = 10\Omega$$

$$I_2 = ?$$

Displacement current

- Ampere's law in static electric field

$$\nabla \times \vec{H} = \vec{J},$$

- Ampere's law in time-varying electric field

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t},$$

- proof of Ampere's law:

$$\nabla \cdot \vec{D} = \rho_v, \quad Q = \int_v (\nabla \cdot \vec{D}) dv = \oint_s D \cdot d\vec{s},$$

$$I = \oint_S \vec{J}' \cdot d\vec{s} = \frac{\partial}{\partial t} Q = \frac{\partial}{\partial t} \int_v (\nabla \cdot \vec{D}) dv = \frac{\partial}{\partial t} \oint_s \vec{D} \cdot d\vec{s} = \oint_s (\frac{\partial}{\partial t} \vec{D}) \cdot d\vec{s},$$

$$\boxed{J' = \frac{\partial \vec{D}}{\partial t}}$$

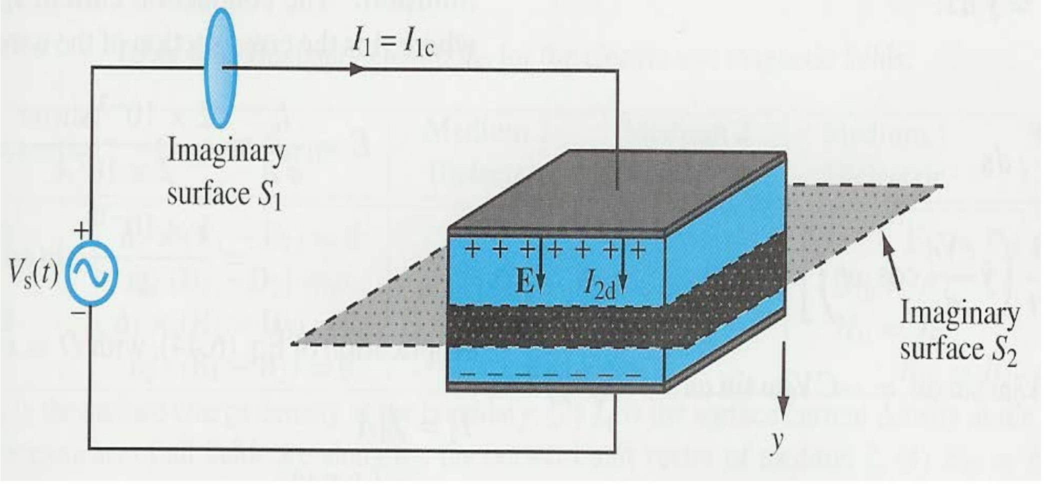
Displacement current density

Displacement current

• Ampere's law in time-varying electric field

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t},$$

Example:



$$V_s = V_0 \cos \omega t,$$

$$I_{1c} = C\frac{dV_C}{dt} = -CV_0\omega \sin \omega t,$$

$$\vec{E} = \hat{y}\frac{V_c}{d} = \hat{y}\frac{V_0}{d}\cos \omega t,$$

$$I_{2d} = \int_S \frac{\partial \vec{D}}{\partial t}\cdot d\vec{s} = -CV_0\omega \sin \omega t,$$

16.360 Lecture 28

- Boundary conditions for Electromagnetic

$$\nabla \cdot \vec{D} = \rho_v,$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$

$$\nabla \cdot \vec{B} = 0,$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t},$$

Maxwell equations

$$D_{1n} - D_{2n} = \rho_v,$$

$$E_{1t} = E_{2t},$$

$$B_{1n} = B_{2n},$$

$$H_{1t} - H_{2t} = J_s,$$

boundary conditions

• <u>Charge-Current continuity Relation</u>

$$I = -\frac{\partial}{\partial t} Q = -\frac{\partial}{\partial t}\int_v \rho_v dv,$$

$$I = \oint_s \vec{J} \cdot d\vec{s} = -\frac{\partial}{\partial t}\int_v \rho_v dv,$$

$$\oint_s \vec{J} \cdot d\vec{s} = \int_v \nabla \cdot \vec{J} = -\frac{\partial}{\partial t}\int_v \rho_v dv,$$

$$\nabla \cdot \vec{J} = -\frac{\partial}{\partial t}\rho_v,$$ charge current continuity equation

$$\oint_s \vec{J} \cdot d\vec{s} = 0,$$ steady state integral form

$$\sum_i I_i = 0,$$ Kirchhoff's current law

• Free-charge dissipation in a conductor

$$\nabla \cdot \vec{J} = -\frac{\partial}{\partial t}\rho_v, \qquad \vec{J} = \sigma E,$$

$$\frac{\sigma}{\varepsilon}\nabla \cdot \vec{D} = \nabla \cdot \vec{J} = -\frac{\partial}{\partial t}\rho_v,$$

$$\frac{\sigma}{\varepsilon}\rho_v = -\frac{\partial}{\partial t}\rho_v,$$

$$\rho_v = \rho_{vo} e^{-t/\tau_r}, \qquad \tau_r = \frac{\varepsilon}{\sigma},$$

- Electromagnetic Potentials

Electrostatics: $\nabla \times \vec{E} = 0, \quad \vec{E} = -\nabla V,$

$$\nabla \cdot \vec{B} = 0, \quad \vec{B} = \nabla \times \vec{A},$$

Dynamic case:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \nabla \times \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{A}),$$

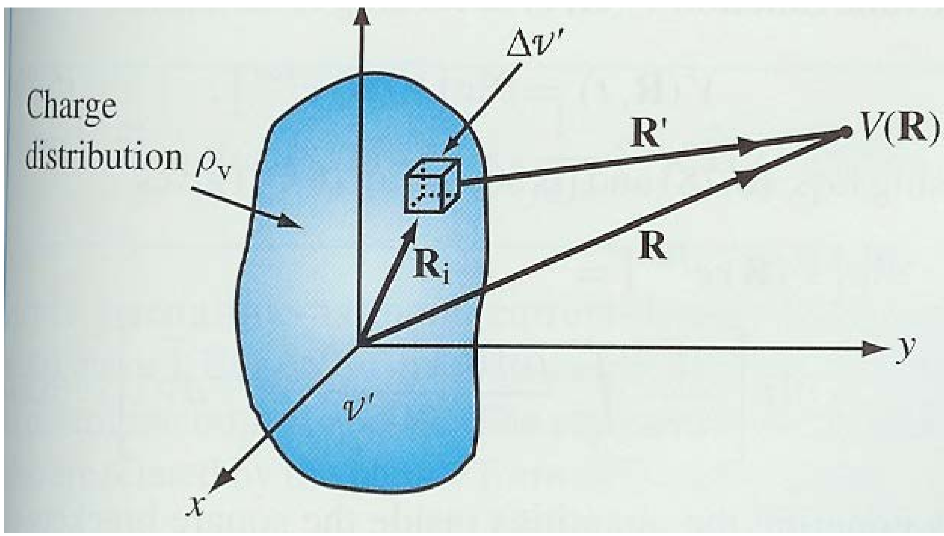
$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t},$$

$$\vec{B} = \nabla \times \vec{A},$$

- Retard Potentials

Electrostatics: $$V(\vec{R}) = \frac{1}{4\pi\varepsilon}\int_{v'} \frac{\rho_{v'(R_i)}}{\vec{R}'} dv',$$

Dynamic case:



$$V(\vec{R},t) \neq \frac{1}{4\pi\varepsilon}\int_{v'} \frac{\rho_{v'}(R_i,t)}{\vec{R}'} dv',$$

$$V(\vec{R},t) = \frac{1}{4\pi\varepsilon}\int_{v'} \frac{\rho_{v'}(R_i,t-R'/u_p)}{\vec{R}'} dv',$$

$$A(\vec{R},t) = \frac{\mu}{4\pi}\int_{v'} \frac{\vec{J}(R_i,t-R'/u_p)}{\vec{R}'} dv',$$

- Time-Harmonic Potentials

$$\rho_{v'}(R_i, t) = \mathrm{Re}\left[\tilde{\rho}_{v'}(R_i) e^{j\omega t}\right],$$

$$\rho_{v'}(R_i, t - R'/u_p) = \mathrm{Re}\left[\tilde{\rho}_{v'}(R_i) e^{j\omega t - j\omega R'/u_p}\right] = \mathrm{Re}\left[\tilde{\rho}_{v'}(R_i) e^{j\omega t - jkR'}\right],$$

$$V(\vec{R}, t) = \mathrm{Re}\left[\frac{1}{4\pi\varepsilon}\int_{v'} \frac{\rho_{v'}(R_i, t - R'/u_p)}{\vec{R}'} dv'\right] = \mathrm{Re}\left[\frac{1}{4\pi\varepsilon}\int_{v'} \frac{\tilde{\rho}_{v'}(R_i) e^{j\omega t} e^{-jkR'}}{\vec{R}'} dv'\right],$$

$$\tilde{V}(\vec{R}) = \frac{1}{4\pi\varepsilon}\int_{v'} \frac{\tilde{\rho}_{v'}(R_i) e^{-jkR'}}{\vec{R}'} dv',$$

$$\tilde{A}(\vec{R}) = \frac{1}{4\pi\varepsilon}\int_{v'} \frac{\tilde{J}_{v'}(R_i) e^{-jkR'}}{\vec{R}'} dv',$$

- Time-Harmonic Potentials

$$\nabla \times \tilde{\vec{H}} = j\omega\varepsilon \tilde{\vec{E}}, \quad \nabla \times \tilde{\vec{E}} = -j\omega\mu \tilde{\vec{H}},$$

$$\nabla \times (\nabla \times \tilde{\vec{E}}) = \nabla(\nabla \cdot E) - \nabla^2 E = -\omega^2 \mu\varepsilon \tilde{\vec{E}},$$

$\nabla \cdot E = 0,$ if no free charge, trans-wave, why?

$$\nabla^2 \tilde{\vec{E}} + \omega^2 \mu\varepsilon \tilde{\vec{E}} = 0, \quad \tilde{\vec{E}}(\vec{R}') = \tilde{\vec{E}}_0 e^{-jkR'}, \quad k^2 = \omega^2 \mu\varepsilon,$$

- example

$\vec{E}(z,t) = \hat{x} 10 \sin(10^{10} t - kz),$ find k?