**Problem 4.22**  Given the electric flux density

\[ \mathbf{D} = \hat{x}2(x + y) + \hat{y}(3x - 2y) \quad (\text{C/m}^2) \]

determine

(a) \( \rho_v \) by applying Eq. (4.26).

(b) The total charge \( Q \) enclosed in a cube 2 m on a side, located in the first octant with three of its sides coincident with the \( x-, y-, \) and \( z- \) axes and one of its corners at the origin.

(c) The total charge \( Q \) in the cube, obtained by applying Eq. (4.29).

**Solution:**

(a) By applying Eq. (4.26)

\[ \rho_v = \nabla \cdot \mathbf{D} = \frac{\partial}{\partial x} (2x + 2y) + \frac{\partial}{\partial y} (3x - 2y) = 0. \]

(b) Integrate the charge density over the volume as in Eq. (4.27):

\[ Q = \int \int \int \nabla \cdot \mathbf{D} \, dV = \int_{x=0}^{2} \int_{y=0}^{2} \int_{z=0}^{2} 0 \, dx \, dy \, dz = 0. \]

(c) Apply Gauss’ law to calculate the total charge from Eq. (4.29)

\[ Q = \oint \mathbf{D} \cdot d\mathbf{s} = F_{\text{front}} + F_{\text{back}} + F_{\text{right}} + F_{\text{left}} + F_{\text{top}} + F_{\text{bottom}}. \]

\[ F_{\text{front}} = \int_{y=0}^{2} \int_{z=0}^{2} (\hat{x}2(x + y) + \hat{y}(3x - 2y)) \bigg|_{x=2} \cdot (\hat{x} \, dz \, dy) \]

\[ = \int_{y=0}^{2} \int_{z=0}^{2} 2(x + y) \bigg|_{x=2} \, dz \, dy = \left( 2z \left( 2y + \frac{1}{2}y^2 \right) \right)_{z=0}^{2} = 24, \]

\[ F_{\text{back}} = \int_{y=0}^{2} \int_{z=0}^{2} (\hat{x}2(x + y) + \hat{y}(3x - 2y)) \bigg|_{x=0} \cdot (\hat{x} \, dz \, dy) \]

\[ = -\int_{y=0}^{2} \int_{z=0}^{2} 2(x + y) \bigg|_{x=0} \, dz \, dy = -\left( zy^2 \right)_{z=0}^{2} = -8, \]

\[ F_{\text{right}} = \int_{x=0}^{2} \int_{z=0}^{2} (\hat{x}2(x + y) + \hat{y}(3x - 2y)) \bigg|_{y=2} \cdot (\hat{y} \, dz \, dx) \]

\[ = \int_{x=0}^{2} \int_{z=0}^{2} (3x - 2y) \bigg|_{y=2} \, dz \, dx = \left( z \left( \frac{3}{2}x^2 - 4x \right) \right)_{z=0}^{2} = -4, \]
\[ F_{\text{left}} = \int_{x=0}^{2} \int_{z=0}^{2} \left( \mathbf{\hat{x}}(x + y) + \mathbf{\hat{y}}(3x - 2y) \right) \bigg|_{y=0} \cdot (-\mathbf{\hat{y}} \, dz \, dx) \]
\[ = -\int_{x=0}^{2} \int_{z=0}^{2} (3x - 2y) \bigg|_{y=0} \, dz \, dx = -\left( z \left( \frac{3}{2}x^2 \right) \right|_{z=0}^{2} - z \left( \frac{3}{2}x^2 \right) \bigg|_{z=0}^{2} = -12, \]

\[ F_{\text{top}} = \int_{x=0}^{2} \int_{z=0}^{2} \left( \mathbf{\hat{x}}(x + y) + \mathbf{\hat{y}}(3x - 2y) \right) \bigg|_{z=2} \cdot (\mathbf{\hat{z}} \, dy \, dx) \]
\[ = \int_{x=0}^{2} \int_{z=0}^{2} 0 \bigg|_{z=2} \, dy \, dx = 0, \]

\[ F_{\text{bottom}} = \int_{x=0}^{2} \int_{z=0}^{2} \left( \mathbf{\hat{x}}(x + y) + \mathbf{\hat{y}}(3x - 2y) \right) \bigg|_{z=0} \cdot (\mathbf{\hat{z}} \, dy \, dx) \]
\[ = \int_{x=0}^{2} \int_{z=0}^{2} 0 \bigg|_{z=0} \, dy \, dx = 0. \]

Thus \( Q = \int \mathbf{D} \cdot ds = 24 - 8 - 4 - 12 + 0 + 0 = 0. \)
**Problem 4.25** The electric flux density inside a dielectric sphere of radius $a$ centered at the origin is given by

$$D = \hat{R} \rho_0 R \quad \text{(C/m}^2\text{)}$$

where $\rho_0$ is a constant. Find the total charge inside the sphere.

**Solution:**

$$Q = \oint_S D \cdot ds = \int_0^\pi \int_0^{2\pi} \hat{R} \rho_0 R \cdot \hat{R} R^2 \sin \theta \ d\theta \ d\phi \bigg|_{R=0}^{R=a}$$

$$= 2\pi \rho_0 a^3 \int_0^{\pi} \sin \theta \ d\theta = -2\pi \rho_0 a^3 \cos \theta \bigg|_0^{\pi} = 4\pi \rho_0 a^3 \quad \text{(C).}$$
Problem 4.29  A spherical shell with outer radius $b$ surrounds a charge-free cavity of radius $a < b$ (Fig. P4.29). If the shell contains a charge density given by

$$\rho_v = -\frac{\rho_{v0}}{R^2}, \quad a \leq R \leq b,$$

where $\rho_{v0}$ is a positive constant, determine $\mathbf{D}$ in all regions.

**Figure P4.29:** Problem 4.29.

**Solution:** Symmetry dictates that $\mathbf{D}$ is radially oriented. Thus,

$$\mathbf{D} = \hat{R} D_R.$$

At any $R$, Gauss’s law gives

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \quad \text{and} \quad \oint_S \hat{R} D_R \cdot \hat{R} d\mathbf{s} = Q,$$

$$4\pi R^2 D_R = Q$$

$$D_R = \frac{Q}{4\pi R^2}.$$

(a) For $R < a$, no charge is contained in the cavity. Hence, $Q = 0$, and

$$D_R = 0, \quad R \leq a.$$

(b) For $a \leq R \leq b$,

$$Q = \int_{R=a}^R \rho_v \, dV = \int_{R=a}^b \frac{-\rho_{v0}}{R^2} \cdot 4\pi R^2 \, dR$$

$$= -4\pi \rho_{v0}(R - a).$$
Hence,

\[ D_R = -\frac{\rho \alpha (R - a)}{R^2}, \quad a \leq R \leq b. \]

\[(c)\] For \( R \geq b, \)

\[ Q = \int_{R=a}^b \rho \nu \ dV = -4\pi \rho \alpha (b - a) \]

\[ D_R = -\frac{\rho \alpha (b - a)}{R^2}, \quad R \geq b. \]