Section 1.3 Predicates and Quantifiers

Assume universe of discourse is all the people who are participating in this course. Also let us assume that we know each person in the course. Consider the following statement: She/he is over 6 feet tall. This statement is not a proposition since we cannot say that it either true or false until we replace the variable (she/he) by a person’s name. The statement “She/he is over 6 feet tall” may be denoted by the symbol P(n) where n stands for the variable and P, the predicate, “is over six feet tall”. The symbol P is used because once the variable is replaced (by a person’s name in this case) the above statement is a proposition. For example if we know that Jim is over 6 feet tall the statement “Jim is over six feet tall” is a (true) proposition.

Another example, For all real numbers x, $x^2 -5x + 6 = (x - 2) (x – 3)$. We could let Q(x) stand for $x^2 -5x + 6 = (x - 2) (x – 3)$. We also note that the truth values of $Q(x)$ is indeed all real numbers.

Quantifiers

There are two quantifiers used in mathematics “for all” and “there exists”. The symbol used “for all” is an upside down A, namely, $\forall$. The symbol used for “there exists” is a backwards E, namely, $\exists$. We all realize that the standard, every day usage of the English language does not necessarily coincide with the Mathematical usage of English so we have to clarify what we mean by the two quantifiers.

<table>
<thead>
<tr>
<th>$\forall$</th>
<th>For all</th>
<th>For every</th>
<th>For each</th>
<th>For any</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\exists$</td>
<td>There exists at least one</td>
<td>There exists</td>
<td>There is</td>
<td>Some</td>
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The table indicates that the mathematical meaning of the universal quantifier, for all, coincides with our everyday usage of this term. However, the mathematical meaning of the existential quantifier does not. When we use the word “some” in everyday language we ordinarily mean two or more, in mathematics the word some means “at least one” which is true when there is exactly one.

The Negation of “For all “Quantifiers

Consider the statement “All people in this course are over 6 feet tall.” Assume it is false. (I am not over six feet tall) How do we prove it is false? All we have to do is to point to one person to prove the statement is false. That is, all we need to do is give one counterexample. We need only show that there exists at least one person in this class who is not over 6 feet tall. Here is a more formal procedure.

**Example 1.** Let P(n) stand for “people in this course are over 6 feet tall”, then the sentence “All people in this course are over 6 feet tall.” Can be written as $\forall n P(n)$ and its negative, $\neg (\forall n P(n))$ is equivalent to $\exists n (\neg P(n))$.

**Example 2.** How would one prove that the sentence “Every person in this course is over six feet tall and is taking the course C Programming.”
P(n) stand for “people in this course are over 6 feet tall” and let
Q(n) stand for “person n is taking the course C Programming” then the given sentence in
symbolic form is ∀ n(P(n) ∧ Q(n)). Next,
¬(∀ n(P(n) ∧ Q(n))) ⇔ ∃ n(¬(P(n) ∧ Q(n)))
⇔ ∃ n(¬P(n) ∨ ¬Q(n)) Note the use of DeMorgan’s law

Why?

There is a person in this course who is not over six feet tall or who is not taking the
course C Programming. OR

Some people in this course are not over six feet tall or are not taking the course C
Programming

The Negation of “There exists “Quantifiers.

Example 3. How would you prove the statement “There is a person in this course over 7
feet tall.” False?
Convince yourself that ¬(∃ n(P(n))) is equivalent to ∀ n(¬P(n)) and use it to answer the
above.

Can you write the negatives of the following expressions using logic?

1. All people in this course are interesting and informative.

2. Some people in this course are having fun. (Note, the word “some” in
mathematics means there is at least one, or there exists.)

3. All people in this course are over 6 feet tall or wear glasses.

4. For all real numbers x, the equation x^2 + 6x + 5 = 0 is true.

For section 1.4 just concentrate on the following examples in your text: 1-3, 14, 15
Methods of Proving Theorems

Why??

Almost nothing in undergraduate and some graduate mathematics courses creates fear, as when an instructor announces that the students in his/her class will be required to do proofs. Yet, we all prove statements frequently. We may be required to explain some fact to a child, a high school student or to a colleague. Our explanation is going to vary depending on what that person knows. When you have successfully explained the fact you have “proven it”. All a proof is, is a logical, detailed explanation of why a statement is true. Our explanation will vary depending on to whom we are talking. Just like their can are different (correct) directions for going from one location to another there are often times different proofs for the same mathematical statement.

What we try to do in any course we take is to improve on our problem-solving ability. In order to solve a problem we must make sure we have a clear idea of what the problem is, what are the assumptions, that is the “givens” and what are we try to accomplish. This is precisely what we do in proofs. Spending some time thinking about proof techniques will enhance our problem solving techniques.

Another purpose of these notes is to help us to be able to read proofs contained in a textbook that we might be studying. Since a proof is a logical, detailed explanation of a statement, it helps to clear up any ambiguity we may have on the concept. It gets right to “the basics”, the core definitions of what we are studying. If we can read and understand a proof we have mastered the key ideas of that concept.

Methods of Proving If … then … types of Theorems

All theorems in mathematics or any application of mathematics can be express as a (or two) “If … then …” statement(s). Therefore the text and the notes below discuss methods of proving “If … then …” statements.

Example 1. The distributive law for numbers should really read as follows:
If a, b and c are any three real numbers then \( a(b + c) = ab + ac \).

Example 2. The commutative law for sets ( under the operation of \( \cup \) ) should be stated as:
If A and B are any two sets then \( A \cup B = B \cup A \).

Example 3. The basic matrix algebra law \( A(BC) = (AB)C \) should really read as follows:
If A, B and C are matrices whose entries are real numbers of orders m x n, n x p and p x l respectively then \( A(BC) = (AB)C \). This example will make more sense after you have studied matrices.

You should be able to take each of the logic and set laws given in the notes and write them in If … then … format.
The Direct Method of proof of $P \Rightarrow C$ Statements.

As we know from our discussions in logic If $P$ then $C$ statements or $P$ implies $C$ statements are frequently written in logical notation as $P \Rightarrow C$.

Recall that $P$ stands for the premise (or hypothesis) of the theorem and $C$ stands for the conclusion. In example 4, part (a) below the premise, $P$, is “$ab = 0$”. The conclusion $C$ is “$a = 0$ or $b = 0$”. What are $P$ and $C$ for examples 2 and 3 and example 4 part (b)? The Direct Method of proof of $P \Rightarrow C$ statements is based on the definition of $P \Rightarrow C$. Look at the two cases where $P$ is true in the definition. If $P$ is true then the complete statement $P \Rightarrow C$ is true when $C$ is true. So to prove a statement of the type $P \Rightarrow C$ true assume $P$ is true and prove $C$ is true. This is called the direct method of proof of a $P \Rightarrow C$ statement. So to use the direct method we start with the assumption $P$ is true and we show that it (naturally) follows that $C$ is true.

Example 2. Prove the following: If $x$ is any even integer and if $y$ is any even integer then $xy$ is an even integer.

Commentary. First, I will comment on some key ideas in proof techniques and then proceed with the proof.

- Most proofs depend on basic definitions and concepts. Do not proceed any further until you truly understand these essential ideas. A major reason for wanting to prove something is because it will force us to learn these key concepts. (Certainly, people probably smarter than us did the proof of this statement many years ago.) In this example the key definition is that of an even integer. What is the definition of an even integer? I am not asking what we think the definition may be but what is the definition. We may have to look it up. Here is a form of the definition of even integer. An even integer is any integer that can be expressed in the form $2n$ for some integer $n$.
- Next do you truly understand the definition? Can you give some examples and explain them. For example 6 is an even integer. Why? Is 0 an even integer? Why? (Answer, yes, because $0 = 2(0)$, that is, 0 can be written as 2 times an integer.)
- What method of proof will you try? Usually try the direct method first.
- What is the process for this method? What is the assumption, the premise? What are you trying to prove? Write the assumption(s) and the conclusion(s) out. This will focus you attention on what you are trying to do.
- Give reasons for each step in your proof. If you cannot give a reason for a particular step it is most likely wrong.
- Relax, with practice you will get most proofs most of the time.

Now I will repeat the statement and proof it. Before you look at my proof try one of your own.

If $x$ is any even integer and if $y$ is any even integer then $xy$ is an even integer.
Proof: (Direct)

Assume $x$ and $y$ are even integers is true. Start with this.

To prove $xy$ is an even integer is true.
Here write down specifically what you want to do. Here you want to prove that $xy$ is an even integer so you must show that $xy$ can be expressed as 2 times some integer. End with this.

$x$ and $y$ are even integers implies that $x = 2n$ and $y = 2m$ for some integers $n$ and $m$. Why?

- this implies that $xy = (2n)(2m)$ Why?
- this implies that $xy = 2(2nm)$ Why?
- this implies that $xy$ is an even integer. Why?

DONE.

Some people like to use logical notation so they would write the above as:

$x$ and $y$ are even integers $\Rightarrow x = 2n$ and $y = 2m$ for some integers $n$ and $m$. Why?

$\Rightarrow xy = (2n)(2m)$ Why?
$\Rightarrow xy = 2(2nm)$ Why?
$\Rightarrow xy$ is an even integer. Why?

DONE.

Example 3. Prove the following: Let $a$, $b$, and $c$ be integers. Then if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.

Comments.

- The sentence “Let $a$, $b$, and $c$ be integers.” is just another way of saying let $a$, $b$, and $c$ be any integers or for all integers $a$, $b$, and $c$. The statement we are trying to prove is “if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$”. What is the premise? What is the conclusion?
- Before we begin we need to understand what $x \mid y$ means. Assume $x$ and $y$ are integers where $x \neq 0$. Read $x \mid y$ as $x$ “divides” $y$, which means $x$ divides evenly into $y$, no remainder. So, $2 \mid 8$ and $3 \mid 36$ but $2$ does not divide $5$. Therefore, $x \mid y$ means $\frac{y}{x} = $ some integer, say $k$, that is $y = kx$. A formal definition follows.
- **Definition.** Let $x$ and $y$ be integers with $x \neq 0$. We say that $x$ divides $y$ if there exists an integer $k$ such that $y = kx$.

Let $a$, $b$, and $c$ be integers. Then if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.

Proof: (Direct)

Assume $a \mid b$ and $a \mid c$ (is true).

To prove $a \mid (b + c)$ (is true). That is we want to prove that $(b + c) = a \cdot $ (some integer).
a | b and a | c ⇒ b = ak for some integer k and
c = al for some integer l. This is the definition of divides.
⇒ b + c = ak + al Simply add both sides of the above equations.
⇒ b + c = a(k + l) The distributive law.
⇒ b + c = a(some integer) The sum of 2 integers is an integer.
⇒ a |(b + c) This is the definition of divides.

Exercises
Prove the following:
1. If x is any even integer and if y is any even integer then x + y is an even integer.
2. If x is any odd integer and if y is any odd integer then xy is an odd integer.
3. Let a, b, and c be any integers. If a | b, then a | bc.
4. Let a, b, and c be any integers. If a | b and b | c, then a | c.
5. Let a, b, and c be any integers. If ac | bc, then a | b.
6. If a is an odd integer then 8|(a² – 1)

How to prove iff Theorems.

From the material in logic we know that “iff” means “if and only if” and the logical
symbol, ↔ or ⇔ is often used in place of iff. From the chart Common Implications and
Equivalences we know that
(p ↔ q) ⇔ [(p → q) ∧ (q → p)] is a tautology, a statement, which is always true. That is in
words:
p iff q is equivalent to saying if p then q and if q then p or
p iff q is equivalent to saying if p then q and conversely.

That is, to prove an iff theorem one must prove two if … then … theorems.

Example 4. Let a and b be any two real numbers. Prove the following: ab = 0 iff a = 0
or b = 0.
Proof: We must prove the following two statements:
(a) If ab = 0 then a = 0 or b = 0 and
(b) If a = 0 or b = 0 then ab = 0.

To prove part (a) we use the direct method of proof.
Assume that ab = 0 and prove that a = 0 or b = 0. Why?
To prove a = 0 or b = 0 we need only show that one of the two parts of this or statement
is true. Why? To do this assume that a ≠ 0 show that when this happens b must be equal
to 0. Why does this work?

Since a ≠ 0 then 1/a exists. Why? Now multiply both sides of ab = 0 by 1/a to obtain b =
0. This proves part (a).

To prove part (b) we use the direct method of proof.
Assume a = 0 or b = 0
To prove ab = 0
Here we have to show that whether we assume that \( a = 0 \) or we assume that \( b = 0 \) the conclusion, \( ab = 0 \) follows. So we have two cases/situations.

Case I. Assume that \( a = 0 \) and prove that \( ab = 0 \). This follows from the definition of the multiplication of a number by 0 that we all learned in elementary algebra.

Case II. Assume that \( b = 0 \) and prove that \( ab = 0 \). Again, this follows from the definition of the multiplication of a number by 0.

Now do example 24 of your text.

Here are some additional exercises.

1. Prove the following: The equation \( ax^2 + bx + c = 0 \) has a unique solution if and only if the discriminant \( b^2 - 4ac = 0 \). Hint: First express the iff statement as two if … then… statements and then prove each using the direct method of proof. For each if … then … statement carefully write down your assumption and what you are trying to prove.

2. Let \( n \) be a positive integer. Prove: \( n \) is odd iff \( 5n + 6 \) is odd.

The “Proof by Contradiction” method of proof of \( P \Rightarrow C \) Statements.

An If … then … (that is an \( P \Rightarrow C \)) statement is either true or false. If a \( P \Rightarrow C \) statement is not false then it must be true. So if we assume that a \( P \Rightarrow C \) statement is false and find that this cannot happen (get a contradiction) then the given \( P \Rightarrow C \) statement must be true. In the following identity I will use the conventional \( P \rightarrow C \) in place of \( P \Rightarrow C \).

The proof by contradiction method of proof is based on the identity:

\[
\neg[P \rightarrow C] \iff [P \land \neg C].
\]

How? So to use the “proof by contradiction” method of proof on a \( P \Rightarrow C \) statement, assume \( P \) is true and \( C \) is false and prove that this gives a contradiction. Hint: The usual place to start using this method is generally the false part of the “assumption” that \( C \) is false. Why?

Example 5. Use the “proof by contradiction” method of proof to prove:

If \( x \) and \( y \) are any real numbers such that \( x + y \leq \frac{1}{2} \), then either \( x \leq \frac{1}{2} \) or \( y \leq \frac{1}{2} \).

Explain the procedure and all steps.

Proof: (by the proof by contradiction method) Assume that \( x + y \leq \frac{1}{2} \) is true and that \( x \leq \frac{1}{2} \) or \( y \leq \frac{1}{2} \) is false, and show that this leads to a contradiction. Where do we start?
$x \leq \frac{1}{2}$ or $y \leq \frac{1}{2}$ is false means that not ($x \leq \frac{1}{2}$ or $y \leq \frac{1}{2}$) is true, that is, that ($x > \frac{1}{2}$ and $y > \frac{1}{2}$) is true. Why? But ($x > \frac{1}{2}$ and $y > \frac{1}{2}$) is true means that $x + y > 1$. This a contradiction to the assumption that $x + y \leq \frac{1}{2}$.

More exercises

Use the proof by contradiction method of proof to prove the following:

3. If $n^2$ is an even integer then $n$ is an even integer.
4. If $n$ is an integer and $3n + 2$ is even then $n$ is even.
5. If $n^2$ is not divisible by 3, then $n$ is not divisible by 3.
6. If $xy$ is an odd integer then (both) $x$ and $y$ are odd integers. Some hints for this one. When you assume that $x$ and $y$ are odd integers is false this means (by DeMorgan’s law). This or, in your assumption $x$ is even or $y$ is even means you have to prove that the conclusion ($xy$ is odd) using 3 cases (or situations), namely, Case 1. $x$ even, $y$ odd, Case 2. $x$ odd, $y$ even and Case 3. $x$ even and $y$ even.

How do we prove a statement is false?

Finally how does one show/prove that a statement is false? Consider the following sentence. Every person taking this course is over six feet tall. To prove this statement is incorrect we need only show that there is at least one person taking the course who is not over six feet tall. This is called giving a counterexample (to the given statement). Look up this term in the index of the text for more examples.

The negation of a “for all” statement is that there exists (there is at least one) example where the statement is not true. That is, in notation $\neg(\forall x P(x)) \iff \exists x (\neg P(x))$.

Warning. (Counter) examples are used to prove theorems/statements false but examples cannot be used to prove theorems true.

Example 6. How does one prove that the following is false? Let $A$, $B$, and $C$ be any three sets.
If $A \cup C = B \cup C$ then $A = B$.

First one must realize that this statement really says (as do many statements in math) For all sets $A$, $B$, and $C$ the above statement is true. So this is a “for all” statement. So the above discussion says that we must find a concrete example where the given If … then … statement is not true. How do we do this? When is an If… then… statement false? This we know (from the definition of the conditional, see page 5, table 5) is when the premise is true and the conclusion is false. So we want an example of a set $A$, a set $B$ and a set $C$ where $A \cup C = B \cup C$ is true but $A = B$ is false. Can you produce such an example?
Skip examples 7 and 8 until you have studied matrices.

**Example 7.** Let A and B be \( n \times n \) matrices, whose entries are real numbers. If \( AB = 0 \) then \( A = 0 \) or \( B = 0 \).

(Again, what this means is for all \( n \times n \) matrices A and B the following is true: If \( AB = 0 \) then \( A = 0 \) or \( B = 0 \).) To prove that this statement is false, all you need to do is to find one example (a \( 2 \times 2 \) matrices will do) where the premise \( AB = 0 \) is true and the conclusion \( A = 0 \) or \( B = 0 \) is false. Can you do this? Hint, when is an "or" statement false? Look at the logic laws.

**Example 8.** Let A and B be \( n \times n \) matrices. Determine if any part of the statement: \( AB = 0 \) iff \( A = 0 \) or \( B = 0 \) is true. Left up to the student.

**Example 9.** Is the following true or false?
For all real numbers \( x \) and \( y \), \( \lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil \). Note this sentence can also be expressed as: If \( x \) and \( y \) are any two real numbers then \( \lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil \). Left up to the student.

Some “do’s” and “don’ts” for proofs.

1. Always try the direct method of proof first. It usually works.
2. Always outline carefully what your assumptions are and what you are trying to prove. This will focus your attention on what you are given and where you are going.
3. Clearly specify the method of proof you are using, again this will focus your attention on the correct process.
4. Keep in mind that a proof is only a detailed explanation. If you don’t understand what you have written then it is most likely wrong.
5. Above all else, RELAX; the worst that can happen is that you cannot complete the proof. That’s OK, hopefully by writing down part of the proof you have focused you attention on the basic definitions and or concepts that you are studying.
6. If you are stuck partly through a proof, again that’s OK; in fact, it is good to realize that you are stuck. Try another approach.

Can you prove the following:

10. \( x \) is an odd integer iff \( x^2 + 2x + 1 \) is even

In section 1.5 of the text skip to the material titled **Methods of Proving Theorems**. Read this material carefully. Concentrate on the examples and the two principle methods of proving \( \text{If} \ldots \text{then} \ldots \) theorems. These methods are: the **direct method** and the **proof by contradiction** method. After you have studied the material in the text continue to read the following. Since any if \( \ldots \) then \( \ldots \) statement is equivalent to its contrapositive another approach would be to write the contrapositive of the given statement and to prove it using one of the above two methods. **Warning**, the literature is a little confusing.
concerning the names of the above two methods. In some texts the proof by contradiction method is called the indirect method. I will stick to the terms used in our text.