Functions

For section 1.8 carefully read all the material and do all the assigned exercises.

Introductory Remarks

When you first studied functions you used notation like: \( y = 2x + 3 \) or \( f(x) = 2x + 3 \) to describe functions. You probably, assumed correctly that you could replace \( x \) by any real number which made sense in the equation. In many applications of functions it is very important to be more precise in our description of functions. For example, in a particular application the replacement of \( x \) in \( f(x) = 2x + 3 \) by negative numbers may not make any sense at all. The variable \( x \) could represent sales of a product or population in a town etc. The set of values one is allowed to use for \( x \) in \( f(x) \) is called the domain of the function. So we need to define the domain of a function more accurately. Another illustration, many of you may be familiar with functions, which “undo what the original function did”. Think of \( x^3 \) and \( \sqrt{x} \) (or \( \sin x \) and \( \sin^{-1} x \)). What does the INV or \( 2^\text{nd} \) function key on many calculators accomplish? Given any function, can we always “reverse the process”? What is this “reverse process” called? In this section the author will develop the language of functions so that these questions and others can be answered. To accomplish this the author wishes to achieve several goals:

1. Give a more precise definition of the term function and some of the terminology of functions. What is the domain? What is the codomain? What is the range or image of the function? What is a one-to-one function? Onto function? What is the composition of two functions?
2. Give examples, which are helpful in the applications of discrete mathematics.

Study the definitions of the text and first think in terms of examples whose domain and codomain are finite sets like those in example 6 of the text or those below. As usual in this course you must study the basic definitions carefully before you proceed with the examples.

Example 1. A nice example that I found that helps people to visualize the different definitions encountered in the text is the following: Think of a classroom, a conference room, an auditorium etc.

Let \( A = \{ \text{of all people in this room} \} \). Let \( B = \{ \text{of all chairs in this room} \} \). Note assume no one is standing. Define the function \( g \) from the set \( A \) to the set \( B \) (Notation \( g: A \longrightarrow B \)) by the rule each person is assigned the chair she/he is sitting in.

(a) Is this function \( g \) one-to-one? What would the seating arrangement be if the function were not one-to-one? Hint: one form of the definition of one-to-one function is: If \( x \neq y \) then \( f(x) \neq f(y) \) for all \( x \) and \( y \) in the domain of the function.

(b) Is your function \( g \) onto? What would the seating arrangement be if the function were not onto?
Example 2. $A = \{1,2,3\}$ I claim there are six one-to-one functions, $f$, from $A$ to $A$. 
(Noteation, $f: A \longrightarrow A$). Use the Rule of Products to prove this. Can you list all six one-to-one functions? Are all six of your functions onto? Why? Can you write the inverse function for each of the above six functions?

Example 3. $A = \{1,2,3\}$ and let $B = \{a, b, c, d\}$. I claim that there are 24 one-to-one functions from $A$ to $B$ for this example. Can you list several examples of one-to-one functions from $A$ to $B$? Can you prove that there are 24 of them? Hint: Do NOT simply list them, but use the Rule of Products. Are there any onto functions from this set $A$ to the given set $B$? Why? Give one example of a 1-1 function from $A$ to $B$. Does this example have an inverse? Why not? Does any 1-1 function from this set $A$ to this set $B$ have an inverse? Why not?

Now read the text and do the exercises.