Matrix Algebra

Before reading the text or the following notes glance at the following list of basic matrix algebra laws.

Some Basic Matrix Laws

Assume the orders of the matrices are such that the following make sense. What should the orders be? How about the entries of the matrices? After you read the text and the notes you should be able to answer these questions.

\[
\begin{align*}
(1) & \quad A + B = B + A \\
(2) & \quad A + (B + C) = (A + B) + C \\
(3) & \quad c(A + B) = cA + cB, \text{ where } c \in \mathbb{R} \\
(4) & \quad (c_1 + c_2)A = c_1A + c_2A, \text{ where } c_1, c_2 \in \mathbb{R} \\
(5) & \quad c_1(c_2A) = (c_1c_2)A, \text{ where } c_1, c_2 \in \mathbb{R} \\
& \quad \text{and (5a) } cA = Ac, \text{ where } c \in \mathbb{R} \\
(6) & \quad 0A = 0, \text{ where } 0 \text{ is the zero matrix} \\
(7) & \quad 0A = 0, \text{ where } 0 \text{ on the left is the number zero} \\
(8) & \quad A + 0 = A \\
(9) & \quad A + (-1)A = 0 \\
(10) & \quad A(B + C) = AB + AC \\
(11) & \quad (B + C)A = BA + CA \\
(12) & \quad A(BC) = (AB)C \\
(13) & \quad IA = A \text{ and } AI = A \\
(14) & \quad \text{If } A^{-1} \text{ exist, } (A^{-1})^{-1} = A \\
(15) & \quad \text{If } A^{-1} \text{ and } B^{-1} \text{ exist, } (AB)^{-1} = B^{-1}A^{-1}
\end{align*}
\]

Even if you have not studied matrix algebra you should recognize that these laws resemble those of high school algebra. Matrix algebra behaves much like high school algebra. There are some similarities and some differences. The key difference is that the commutative law under multiplication (i.e. \(ab = ba\)), which is true for high school algebra, is not true for matrix algebra. More about this later.

You should first read the text on matrix algebra and then read the following notes, which amplifies the material on matrix algebra. Try several problems at the end of the notes where further practice and basic understanding of the material is needed.

A few comments on the text and note material.

1. Matrices may also be denoted by parentheses.

2. Entries of a matrix may be any elements, for example: real numbers, complex numbers, people’s names (think of a seating chart in a classroom whose desks are fixed on the floor so in a seating chart we could indicate the name of the person seating in that chair and if the chair is empty we could simply denote that fact by the word empty in the appropriate location). Another example, in
the previous seating chart example use a 1 (or yes) if the seat is occupied and a 0 (or no ) if the seat is unoccupied.

3. If the entries of the matrix are real numbers we would suspect that the arithmetic of real numbers would help us define the operations of addition and multiplication on matrices. If the entries of the matrix are 0’s and 1’s (false and true) then we are “living in the world” of logic so we suspect we would use the arithmetic of logic as our underlining arithmetic for matrices. Recall that the arithmetic of logic is frequently called Boolean Arithmetic. Further, recall that the OR or \( \lor \) operation is written as simply + but still read as “or” and the AND or \( \land \) operation is written as \( \cdot \) but still read as “and”. The text in definition 8 defines component-wise \( \lor \). This is frequently written as + as stated above. The text also defines component-wise \( \land \). This operation is not defined in many texts. When it is, it is denoted by the symbol \( \land \). Definition 9 of the text defines the “multiplication” of matrices, which most closely resembles the “regular” multiplication of matrices which many texts write using the usual multiplication symbol of \( \cdot \), where our text uses the symbol, \( \circ \).

For week six do the assignment in the syllabus for the text.

Once you have completed this assignment study the notes that follow. In the notes that follow there are numerous additional examples and clarifications on the text material that you may find helpful. In addition, there are concepts covered in the notes not discussed in the text.
The following notes (taken from a text I coauthored with K. Levasseur) contain additional examples and concepts that you might find helpful. If you need more practice in matrix algebra try the following: The solutions are at the end of the notes.

Sections 5.1–5.3 do the odd numbered problems.
Section 5.4 numbers 1, 2, 3, 4, 5
Section 5.4 numbers 1, 3, 4
Section 5.6 numbers 1, 3, 5

5 GOALS
The purpose of this chapter is to introduce you to matrix algebra, which has many applications. You are already familiar with several algebras: elementary algebra, the algebra of logic, the algebra of sets. We hope that as you studied the algebra of logic and the algebra of sets, you compared them with elementary algebra and noted that the basic laws of each are similar. We will see that matrix algebra is also similar. As in previous discussions, we begin by defining the objects in question and the basic operations.

5.1 Basic Definitions
Definition: Matrix. A matrix is a rectangular array of elements of the form

\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & \ldots & A_{1n} \\
A_{21} & A_{22} & A_{23} & \ldots & A_{2n} \\
A_{31} & A_{32} & A_{33} & \ldots & A_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_{m1} & A_{m2} & A_{m3} & \ldots & A_{mn}
\end{bmatrix}
\]

A convenient way of describing a matrix in general is to designate each entry via its position in the array. That is, the entry \(A_{44}\) is the entry in the third row and fourth column of the matrix \(A\). Since it is rather cumbersome to write out the large rectangular array above each time we wish to discuss the generalized form of a matrix, it is common practice to replace the above by
A = \{ A_{ij} \}. We will assume that each entry \( A_{ij} (1 \leq i \leq m, 1 \leq j \leq n) \) is a real number. However, entries can come from any set, for example, the set of complex numbers.

**NOTE:** Many texts use lower case letters so the entry \( A_{ij} \) is written as \( a_{ij} \).

**Definition: Order.** The matrix \( A \) above has \( m \) rows and \( n \) columns. It therefore, is called an \( m \times n \) (read "m by n") matrix, and is said to be of order \( m \times n \).

We will use \( M_{m \times n} (R) \) to stand for the set of all \( m \times n \) matrices whose entries are real numbers.

**Example 5.1.1.**

\[
A = \begin{bmatrix} 2 & 3 \\ 0 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}, \quad \text{and } D = \begin{bmatrix} 1 & 2 & 5 \\ 6 & -2 & 3 \\ 4 & 2 & 8 \end{bmatrix}
\]

are \( 2 \times 2, 3 \times 1, \) and \( 3 \times 3 \) matrices respectively.

Since we now understand what a matrix looks like, we are in a position to investigate the operations of matrix algebra for which users have found the most applications.

**Example 5.1.2.** First we ask ourselves: Is the matrix \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \) equal to the matrix \( B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \)? Next, is the matrix \( A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \) equal to the matrix \( B = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \)? Why not? We formalize in the following definition.

**Definition: Equality.** The matrix \( A \) is said to be equal to the matrix \( B \) (written \( A = B \)) if and only if:

1. \( A \) and \( B \) have the same order, and
2. corresponding entries are equal: that is \( A_{ij} = B_{ij} \) for all \( i \) and \( j \).

**5.2 Addition and Scalar Multiplication**

**Example 5.2.1.** Concerning addition, it seems natural that if

\[
A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 4 \\ -5 & 2 \end{bmatrix}, \quad \text{then}
\]

\[
A + B = \begin{bmatrix} 1+3 & 0+4 \\ 2+(-5) & (-1)+2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ -5 & 2 \end{bmatrix}. \quad \text{If, however,}
\]
A. Doerr

\[
A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 0 \\ 2 & 8 \end{bmatrix}
\]
can we find \( A + B \)?

**Definition: Addition.** Let \( A \) and \( B \) be \( m \times n \) matrices. Then \( A + B \) is an \( m \times n \) matrix where \((A + B)_{ij} = A_{ij} + B_{ij}\) (read "the \( i \)th \( j \)th entry of the matrix \( A + B \) is obtained by adding the \( i \)th \( j \)th entry of \( A \) to the \( i \)th \( j \)th entry of \( B \))."

It is clear from Example 5.2.1 and the definition of addition that \( A + B \) is defined if and only if \( A \) and \( B \) are of the same order.

Another frequently used operation is that of multiplying a matrix by a number, commonly called a *scalar*. Unless specified otherwise, we will assume that all scalars are real numbers.

**Example 5.2.2.** If \( c = 3 \) and if \( A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \) and we wish to find \( cA \), it seems natural to multiply each entry of \( A \) by 3 so that

\[
3A = \begin{bmatrix} 3 \cdot 1 & 3 \cdot -2 \\ 3 \cdot 3 & 3 \cdot 5 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 9 & 15 \end{bmatrix}
\]

**Definition: Scalar Multiplication.** Let \( A \) be an \( m \times n \) matrix and \( c \) a scalar. Then \( cA \) is the \( m \times n \) matrix obtained by multiplying \( c \) times each entry of \( A \); that is \((cA)_{ij} = cA_{ij}\).

### 5.3 Multiplication of Matrices

A definition that is more awkward to motivate (and we will not attempt to do so here) is the product of two matrices. The reader will see in further illustrations and concepts that if we define the product of matrices the following way, we will obtain a very useful algebraic system that is quite similar to elementary algebra.

**Definition: Multiplication.** Let \( A \) be an \( m \times n \) matrix and let \( B \) be an \( n \times p \) matrix. Then the product of \( A \) and \( B \), denoted by \( AB \), is an \( m \times p \) matrix whose \( i \)th row \( j \)th column entry

\[
(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \ldots + A_{in}B_{nj}
\]

\[
= \sum_{k=1}^{n} A_{ik}B_{kj}
\]

for \( 1 \leq i \leq m, 1 \leq j \leq p \)

The mechanics of computing one entry in the product of two matrices is illustrated in Figure 5.3.1. The computation of a product can take a considerable
amount of time in comparison to the time required to add two matrices. Suppose that A and B are \( n \times n \) matrices; then \( (AB)_{ij} \) is determined using \( n \) multiplications and \( n - 1 \) additions. The full product takes \( n^3 \) multiplications and \( n^3 - n^2 \) additions. This compares with \( n' \) additions for the sum of two \( n \times n \) matrices. The product of two 10 by 10 matrices will require 1,000 multiplications and 900 additions, clearly a job that you would assign to a computer. The sum of two matrices requires a more modest 100 additions. This analysis is based on the assumption that matrix multiplication will be done using the formula that is given in the definition. There are more advanced methods that, in theory, reduce multiplication times. Strassen's algorithm (see Aho, Hopcroft, and Ullman, 1974) computes the product of two \( n \times n \) matrices using no more than a multiple of \( n^{2.8} \) operations. That is, there exists a number \( C \) (which is large) such that no more than \( C \cdot n^{2.8} \) operations are needed to complete a multiplication. Since \( C \) would be so large, this algorithm is useful only if \( n \) is large.

Example 5.3.1.

Let \( A \) be the 3 \( \times \) 2 matrix
\[
\begin{bmatrix}
1 & 0 \\
3 & 2 \\
-5 & 1
\end{bmatrix}
\]
and \( B \) be the 2 \( \times \) 1 matrix
\[
\begin{bmatrix}
6 \\
1
\end{bmatrix}
\]

Then \( AB \) =
\[
\begin{bmatrix}
1 & 0 \\
3 & 2 \\
-5 & 1
\end{bmatrix}
\begin{bmatrix}
6 \\
1
\end{bmatrix}
\]
$= \text{ the 3 x 1 matrix } \begin{bmatrix} (1)(6) + (0)(1) \\ (3)(6) + (2)(1) \\ (-5)(6) + (1)(1) \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ -29 \end{bmatrix}$

Remarks:

(1) The product $AB$ is defined only if $A$ is an $m \times n$ matrix and $B$ is an $n \times p$ matrix; that is, the two "inner" numbers must be the same. Furthermore, the order of the product matrix $AB$ is the "outer" numbers, in this case $m \times p$.

(2) It is wise to first obtain the order of the product matrix. For example, if $A$ is a $3 \times 2$ matrix and $B$ is a $2 \times 2$ matrix, then $AB$ is a $3 \times 2$ matrix of the form

$$AB = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix}$$

Then to obtain, for example, $C_{31}$, we multiply corresponding entries in the third row of $A$ times the first column of $B$ and add the results.

Example 5.3.2.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$

Then $AB = \begin{bmatrix} (1)(3) + (0)(2) & (1)(0) + (0)(1) \\ (0)(3) + (3)(2) & (0)(0) + (3)(1) \end{bmatrix}$

$= \begin{bmatrix} 3 & 0 \\ 6 & 3 \end{bmatrix}$

Note: $BA = \begin{bmatrix} 3 & 0 \\ 2 & 3 \end{bmatrix} \neq AB$

Remarks:

(1) An $n \times n$ matrix is called a square matrix.
(2) If $A$ is a square matrix, $AA$ is defined and is denoted by $A^2$, and
AAA = A^3. Similarly, AA ... A = A^n, where A is multiplied by itself n times.

(3) The m x n matrices each of whose entries is 0 are denoted by 0_{m x n} or simply 0, when no confusion arises.

EXERCISES FOR SECTIONS 5.1 THROUGH 5.3 (The solutions begin on page 20.)

A Exercises:

1. Let \( A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \), \( B = \begin{bmatrix} 0 & 1 \\ 3 & -5 \end{bmatrix} \) and \( C = \begin{bmatrix} 0 & 1 & -1 \\ 3 & -2 & 2 \end{bmatrix} \).

Determine:
(a) \( AB \) and \( BA \).
(b) \( A + B \) and \( B + A \).
(c) If \( c = 3 \), show that \( c(A + B) = cA + cB \).
(d) Show that \( (AB)C = A(BC) \).
(e) \( A^2C \).
(f) \( B + 0 \).
(g) \( A0_{2x2} \) and \( 0A \), where \( 0 = 0_{2x2} \) is the 2 x 2 zero matrix.
(h) \( 0A \), where 0 is the real number (scalar) zero.
(i) Let \( c = 2 \) and \( d = 3 \). Show that \( (c + d)A = cA + dA \).

2. Let \( A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 5 \\ 3 & 2 & 1 \end{bmatrix} \), \( B = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & 2 \\ -1 & 3 & -2 \end{bmatrix} \), and \( C = \begin{bmatrix} 2 & 1 & 2 & 3 \\ 4 & 0 & 1 & 1 \\ 3 & -1 & 4 & 1 \end{bmatrix} \).

Compute, if possible:
(a) \( A - B \)
(b) \( AB \)
(c) \( AC - BC \) Be efficient, use the distributive law first and save roughly half the computations.
(d) \( A(BC) \)
(e) \( CA - CB \) Compare the solution with part (c). Do the solutions have to be the same?
(f) \( C \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \)

3. Let \( A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \), Find a matrix B such that \( AB = I \) and \( BA = I \), where \( I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \).

4. Find AI and BI where I is as in Exercise 3, and A and B are the following matrices:
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\[ A = \begin{bmatrix} 1 & 8 \\ 9 & 5 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} -2 & 3 \\ 5 & -7 \end{bmatrix} \]

What do you notice?

5. Find \( A^3 \) if \( A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \). What is \( A^{15} \) equal to? What are \( A^3 \) and \( A^{15} \) if \( A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \)?

B Exercises:

6. (a) Determine \( I^2, I^3 \), if \( I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

(b) What is \( I^n \) equal to for any \( n \neq 1 \)?

(c) Prove your answer to part b by induction.

7. (a) If \( A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \), show that \( AX = B \) is a way of expressing the system of equations

\[ 2x_1 + x_2 = 3 \\
1x_1 - 1x_2 = 1 \]

(b) Express the following systems of equations using matrices:

(i) \[ 2x_1 - x_2 = 4 \\
x_1 + x_2 = 0 \]

(ii) \[ x_1 + x_2 + 2x_3 = 1 \\
x_1 + 2x_2 - x_3 = -1 \\
x_1 + 3x_2 + x_3 = 5 \]

(iii) \[ x_1 + x_2 = 3 \\
x_2 = 5 \\
x_1 + 3x_3 = 6 \]
5.4 Special Types of Matrices

We have already investigated one special type of matrix, namely the zero matrix, and found that it behaves in matrix algebra in an analogous fashion to the real number 0; that is, as the additive identity. We will now investigate the properties of a few other special matrices.

Definition: Diagonal Matrix. A square matrix \( D \) is called a diagonal matrix if \( D_{ij} = 0 \) whenever \( i \neq j \).

Example 5.4.1.

\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix}, \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

all diagonal matrices.

In Example 5.4.1, the 3 x 3 diagonal matrix \( I \) whose diagonal entries are all 1s has the singular property that for any other 3 x 3 matrix \( A \) we have \( AI = IA = A \). For example:

Example 5.4.2.

\[
\begin{align*}
\text{If } A &= \begin{bmatrix} 1 & 2 & 5 \\ 6 & 7 & -2 \\ 3 & -3 & 0 \end{bmatrix}, \text{ then} \\
AI &= \begin{bmatrix} 1 & 2 & 5 \\ 6 & 7 & -2 \\ 3 & -3 & 0 \end{bmatrix} \\
IA &= \begin{bmatrix} 1 & 2 & 5 \\ 6 & 7 & -2 \\ 3 & -3 & 0 \end{bmatrix}
\end{align*}
\]

In other words, the matrix \( I \) behaves in matrix algebra like the real number 1; that is, as a multiplicative identity. In matrix algebra the matrix \( I \) is called simply the identity matrix. Convince yourself that if \( A \) is any \( n \times n \) matrix \( AI = IA = A \).

Definition: Identity Matrix. The \( n \times n \) diagonal matrix whose diagonal components are all 1s is called the identity matrix and is denoted by \( I \) or \( I_n \).
In the set of real numbers we realize that, given a nonzero real number \( x \), there exists a real number \( y \) such that \( x \cdot y = y \cdot x = 1 \). We know that real numbers commute under multiplication so that the two equations can be summarized as \( x \cdot y = 1 \). Further we know that \( y = x^{-1} = 1/x \). Do we have an analogous situation in \( M_{n \times n}(\mathbb{R}) \)? Can we define the multiplicative inverse of an \( n \times n \) matrix \( A \)? It seems natural to imitate the definition of multiplicative inverse in the real numbers.

**Definition: Matrix Inverse.** Let \( A \) be an \( n \times n \) matrix. If there exists an \( n \times n \) matrix \( B \) such that \( A \cdot B = B \cdot A = I \), then \( B \) is the multiplicative inverse of \( A \) (called simply the inverse of \( A \)) and is denoted by \( A^{-1} \) (read "\( A \) inverse").

When we are doing computations involving matrices, it would be helpful to know that when we find \( A^{-1} \), the answer we obtain is the only inverse of the given matrix.

Remark: Those unfamiliar with the laws of matrices should go over The proof of Theorem 5.4.1 after they have familiarized themselves with the Basic Matrix Laws on page 1 of the notes of this section.

**Theorem 5.4.1.** The inverse of an \( n \times n \) matrix \( A \), when it exists, is unique.

Proof: Let \( A \) be an \( n \times n \) matrix. Assume to the contrary, that \( A \) has two (different) inverses, say \( B \) and \( C \). Then

\[
B = BI \quad \text{Identity property of } I \\
= B (AC) \quad \text{Assumption that } C \text{ is an inverse of } A \\
= (BA)C \quad \text{Associatively of matrix multiplication} \\
= IC \quad \text{Assumption that } B \text{ is an inverse of } A \\
= C \quad \text{Identity property of } I \\
\]

**Example 5.4.3.** Let \( A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \). Use the definition of matrix inverse to find \( A^{-1} \)? Without too much difficulty, by trial and error, we determine that \( A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \)

If \( A = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix} \) what is \( A^{-1} \)? Here the answer is considerably more difficult. In order to understand more completely the notion of the inverse of a matrix, it would be beneficial to have a formula that would enable us to compute the inverse of at least a \( 2 \times 2 \) matrix. To do this, we need to recall the definition of the determinant of a \( 2 \times 2 \) matrix. Appendix A gives a more
complete description of the determinant of a 2 x 2 and higher-order matrices.

Definition: Determinant (2 x 2 Matrix). Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). The determinant of the matrix \( A \), written det(\( A \)) or \(|A|\), is a real number and is equal to det(\( A \)) = ad - bc.

Example 5.4.4.

If \( A = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix} \), then det(\( A \)) = (1)(5) - (2)(-3)
= 11
If \( A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \), then det(\( A \)) = 0

Theorem 5.4.2. Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). If det(\( A \)) \neq 0, then

\[
A^{-1} = \frac{1}{\text{det}(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

Proof: See Exercise 4.

Example 5.4.5.

If \( A = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix} \), then

\[
A^{-1} = \frac{1}{11} \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 5/11 & -2/11 \\ 3/11 & 1/11 \end{bmatrix}.
\]

The reader should verify that \( AA^{-1} = I \) and \( A^{-1}A = I \).

Example 5.4.6. if \( A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \), then det(\( A \)) = 0, so that 1/det(\( A \)) becomes 1/0, which does not exist, so that \( A^{-1} \) does not exist.

Remarks:

(1) It is clear from Example 5.4.6 and Theorem 5.4.1 that if \( A \) is a 2 x 2 matrix and if det(\( A \)) = 0, then \( A^{-1} \) does not exist.
(2) A formula for the inverse of \( n \times n \) matrices \( n \geq 3 \) can be derived that
also involves \(1/\det(A)\). Hence, in general, if the determinant of a matrix is zero, the matrix does not have an inverse.

(3) Later we will develop a technique to compute the inverse of a higher-order matrix, if it exists.

(4) Matrix inversion comes first in the hierarchy of matrix operations; therefore, \(AB^{-1}\) is \(A(B^{-1})\).

EXERCISES FOR SECTION 5.4 (The solutions begin on page 20.)

A Exercises

1. For the given matrices \(A\) find \(A^{-1}\) if it exists and verify that \(AA^{-1}A^{-1}A = I\). If \(A^{-1}\) does not exist explain why.

(a) \[
A = \begin{bmatrix}
1 & 3 \\
2 & 1
\end{bmatrix}
\]

(c) \[
A = \begin{bmatrix}
1 & -3 \\
0 & 1
\end{bmatrix}
\]

(b) \[
A = \begin{bmatrix}
6 & -3 \\
8 & -4
\end{bmatrix}
\]

(d) \[
A = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

(e) Use the **definition** of the inverse of a matrix to find \(A^{-1}\):

\[
A = \begin{bmatrix}
3 & 0 & 0 \\
0 & 1/2 & 0 \\
0 & 0 & -5
\end{bmatrix}
\]

2. For the given matrices \(A\) find \(A^{-1}\) if it exists and verify that \(AA^{-1}A^{-1}A = I\). If \(A^{-1}\) does not exist, explain why.

(a) \[
A = \begin{bmatrix}
2 & -1 \\
-1 & 2
\end{bmatrix}
\]

(c) \[
A = \begin{bmatrix}
1 & c \\
0 & 1
\end{bmatrix}
\]

(b) \[
A = \begin{bmatrix}
0 & 1 \\
0 & 2
\end{bmatrix}
\]

(d) \[
A = \begin{bmatrix}
a & b \\
b & a
\end{bmatrix}
\]

where \(a > b > 0\).

3. (a) Let \(A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\) and \(B = \begin{bmatrix} 3 & -3 \\ 2 & 1 \end{bmatrix}\)

Verify that \((AB)^{-1} = B^{-1}A^{-1}\).

(b) Let \(A\) and \(B\) be \(n \times n\) invertible matrices. Prove that \((AB)^{-1}\)
B⁻¹A⁻¹. (Hint: Use Theorem 5.4. 1.) Why is the right side of the above statement written "backwards"? Is this necessary?

B Exercises

4. Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). Derive the formula for \( A^{-1} \).

5. (a) Let \( A \) and \( B \) be as in exercise 1a, and show that \( \det(AB) = (\det A)(\det B) \).
(b) It can be shown that the statement in part a is true for all \( n \times n \) matrices. Let \( A \) be any invertible \( n \times n \) matrix. Prove that \( \det(A^{-1}) = (\det A)^{-1} \). Note: \( \det(I) = 1 \). Use the definition of inverse to show that this is so for the \( 2 \times 2 \) matrix \( I \), in fact for the \( n \times n \) matrix \( I \).
(c) Verify that the equation in part b is so for the matrix in Exercise 1a of this section.

6. Prove by induction that \( \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix} \) for \( n \geq 1 \).

7. Prove by induction that if \( n \geq 1 \), \( \det(A^n) = (\det A)^n \).

8. Prove: If the determinant of a matrix \( A \) is zero, then \( A \) does not have an inverse. (Hint: Use the indirect method of proof and Exercise 5.)

C Exercise

9. (a) Let \( A, B, \) and \( D \) be \( n \times n \) matrices. Assume that \( B \) is invertible. If \( A = BDB^{-1} \), prove by induction that \( A^m = B D^m B^{-1} \) is true for \( m \geq 1 \).
(b) Given that \( A = \begin{bmatrix} -8 & 15 \\ -6 & 11 \end{bmatrix} = B \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} B^{-1} \), where \( B = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \), what is \( A^{10} \)?

5.5 Laws of Matrix Algebra

The following is a summary of the basic laws of matrix operations. Assume that the indicated operations are defined; that is, that the orders of the matrices \( A, B, \) and \( C \) are such that the operations make sense.

(1) \( A + B = B + A \)
(2) \( A + (B + C) = (A + B) + C \)
(3) \( c (A + B) = cA + cB \), where \( c \in \mathbb{R} \).
(4) \( (c_1 + c_2)A = c_1A + c_2A \), where \( c_1, c_2 \in \mathbb{R} \).
(5) \( c_1 (c_2 A) = (c_1 c_2)A \), where \( c_1, c_2 \in \mathbb{R} \). Also, \( cA = Ac \).
A. Doerr

(6) \(0A = 0\), where \(0\) is the zero matrix.

(7) \(0A = 0\), where \(0\) on the left is the number 0.

(8) \(A + 0 = A\).

(9) \(A + (-1)A = 0\).

(10) \(A(B + C) = AB + AC\).

(11) \((B + C)A = BA + CA\).

(12) \(A(BC) = (AB)C\).

(13) \(IA = A\) and \(AI = A\).

(14) If \(A^{-1}\) exists, \((A^{-1})^{-1} = A\).

(15) If \(A^{-1}\) and \(B^{-1}\) exist then \((AB)^{-1} = B^{-1}A^{-1}\).

**Example 5.5.1.** If we wished to write out each of the above laws carefully, we would specify the orders of the matrices, For example, law 10 should read:

(10) Let \(A\), \(B\), and \(C\) be \(m \times n\), \(n \times p\), and \(n \times p\) matrices (Why?) respectively. Then \(A(B + C) = AB + AC\).

Remarks:

(1) We notice the absence of the "law" \(AB = BA\). Why?

(2) Is it really necessary to have both a right (No. 11) and a left (No. 10) distributive law? Why?

(3) What does law 8 define? What does Law 9 define?

**EXERCISES FOR SECTION 5.5 (The solutions begin on page 21.)**

A Exercises

1. Rewrite the above laws specifying as in Example 5.5.1 the orders of the matrices.

2. Verify each of the Laws of Matrix Algebra via examples.

3. Let \(A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\), \(B = \begin{bmatrix} 3 & 7 & 6 \\ 2 & -1 & 5 \end{bmatrix}\) and \(C = \begin{bmatrix} 0 & -2 & 4 \\ 7 & 1 & 1 \end{bmatrix}\)

Find:
   (a) \(AB + AC\)  
   (b) \(A^{-1}\)  
   (c) \(A(B + C)\)  
   (d) \((A^2)^{-1}\)

4. Let \(A = \begin{bmatrix} 7 & 4 \\ 2 & 1 \end{bmatrix}\) and \(B = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}\).

Find:
   (a) \(AB\)  
   (b) \(A + B\)  
   (c) \(A^2 + AB + BA + B^2\)  
   (d) \(B^{-1}A^{-1}\)  
   (e) \(A^2 + AB\)
5.6 Matrix Oddities

We have seen that matrix algebra is similar in many ways to elementary algebra. Indeed, if we want to solve the matrix equation $AX = B$ for the variable $X$, we imitate the procedure used in elementary algebra for solving the equation $ax = b$. Notice that the same properties are used in the following detailed solutions of both equations.

<table>
<thead>
<tr>
<th>Solution of $ax = b$</th>
<th>Solution of $AX = B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ax = b$</td>
<td>$AX = B$</td>
</tr>
<tr>
<td>$a^{-1}(ax) = a^{-1}b$ if $a \neq 0$</td>
<td>$A^{-1}(AX) = A^{-1}B$ if $A^{-1}$ exists</td>
</tr>
<tr>
<td>$(a^{-1}a)x = a^{-1}b$</td>
<td>associative law</td>
</tr>
<tr>
<td>$1x = a^{-1}b$</td>
<td>definition of inverse</td>
</tr>
<tr>
<td>$x = a^{-1}b$</td>
<td>definition of identity</td>
</tr>
</tbody>
</table>

Certainly the solution process for $AX = B$ is the same as that of $ax = b$. The solution of $xa = b$ is $x = ba^{-1} = a^{-1}b$. In fact, we usually write the solution of both equations as $x = b/a$. In matrix algebra, the solution of $XA = B$ is $X = BA^{-1}$, which is not necessarily equal to $A^{-1}B$. So in matrix algebra, since the commutative law (under multiplication) is not true, we have to be more careful in the methods we use to solve equations.

It is clear from the above that if we wrote the solution of $AX = B$ as $X = B/A$, we would not know how to interpret the answer $B/A$. Does it mean $A^{-1}B$ or $BA^{-1}$? Because of this, $A^{-1}$ is never written as $1/A$.

Some of the main dissimilarities between matrix algebra and elementary algebra are that in matrix algebra:

1. AB may be different from BA.
2. There exist matrices $A$ and $B$ such that $AB = 0$, and yet $A \neq 0$ and $B \neq 0$.
3. There exist matrices $A$ where $A \neq 0$, and yet $A^2 = 0$.

4. There exist matrices $A$ where $A^2 = A$ with $A \neq 0$ and $A \neq I$.
5. There exist matrices $A$ where $A^2 = I$, where $A \neq I$ and $A \neq -I$.

EXERCISES FOR SECTION 5.6 (The solutions begin on page 22.)

A Exercises

1. Discuss each of the above "oddities" with respect to elementary algebra.

2. Determine $2 \times 2$ matrices which show each of the above "oddities" are true.
B Exercises

3. Prove the following implications, if possible:
   (a) $A^2 = A$ and $\det(A) \neq 0 \Rightarrow A = I$
   (b) $A^2 = I$ and $\det(A) \neq 0 \Rightarrow A = I$ or $A = -I$.

4. Let $M_n$ be the set of real $n \times n$ matrices. Let $P \subseteq M_n$ be the subset of matrices defined by $A \in P$ iff $A^2 = A$. Let $Q \subseteq P$ be defined by $A \in Q$ if and only if $\det A \neq 0$.
   (a) Determine the cardinality of $Q$.
   (b) Consider the special case $n = 2$ and prove that a sufficient condition for $A \in P \subseteq M_2$ is that $A$ has a zero determinant (i.e., $A$ is singular) and $\text{tr}(A) = 1$ where $\text{tr}(A) = a_{11} + a_{22}$ is the sum of the main diagonal elements of $A$.
   (c) Is the condition of part b a necessary condition?

C Exercises

5. Write each of the following systems in the form $AX = B$, and then solve the systems using matrices as was developed at the beginning of section 5.6, that is $X = A^{-1}B$.
   (a) $2x_1 + x_2 = 3$
      $x_1 - x_2 = 1$
   (b) $2x_1 - x_2 = 4$
      $x_1 - x_2 = 0$
   (c) $2x_1 + x_2 = 1$
      $x_1 - x_2 = 1$
   (d) $2x_1 + x_2 = 1$
      $x_1 - x_2 = -1$
   (e) $3x_1 + 2x_2 = 1$
      $6x_1 + 4x_2 = -1$

6. Recall that $p(x) = x^2 - 5x + 6$ is called a polynomial, or more specifically, a polynomial over $\mathbb{R}$, where the coefficients are elements of $\mathbb{R}$ and $x \in \mathbb{R}$. Also, think of the method of solving, and solutions of, $x^2 - 5x + 6 = 0$. We would like to define the analogous situation for $2 \times 2$ matrices. First define where $A$ is a $2 \times 2$ matrix $p(A) = A^2 - 5A + 6I$. Discuss the method of solving and the solutions of $A^2 - 5A + 6I = 0$. 
SUPPLEMENTARY EXERCISES FOR CHAPTER 5
Sections 5.1 through 5.3

1. Determine x and y in the following:
   \[
   \begin{bmatrix}
   x + y & 5 \\
   -2 & x - y - 2
   \end{bmatrix} = \begin{bmatrix}
   3 & 5 \\
   -2 & 4
   \end{bmatrix}
   \]

2. Let
   \[
   A = \begin{bmatrix}
   2 & 1 & 5 \\
   3 & -4 & 1
   \end{bmatrix}, \quad B = \begin{bmatrix}
   3 & -1 & 1 \\
   1 & 2 & -1
   \end{bmatrix}, \quad C = \begin{bmatrix}
   3
   \end{bmatrix}
   \]
   Compute:
   
   (a) 2A - 3B  
   (b) 2A - 5A  
   (c) AC + BC

3. Let A and B be two m x m matrices with AB = BA. Prove by induction on n that AB^n = B^n A for n greater than or equal to 1.

4. Prove by induction that if n is a positive integer, and
   \[
   A = \begin{bmatrix}
   1 & 1 & 0 \\
   0 & 1 & 1 \\
   0 & 0 & 1
   \end{bmatrix}, \quad A^n = \begin{bmatrix}
   1 & n & n(n-1)/2 \\
   0 & 1 & n \\
   0 & 0 & 1
   \end{bmatrix}
   \]

Section 5.4

5. Determine A^{-1} A^3 if A = \begin{bmatrix}
   2 & 3 \\
   1 & 4
   \end{bmatrix}.

6. Let A = \begin{bmatrix}
   4 & -2 \\
   -2 & 5
   \end{bmatrix} and B = \begin{bmatrix}
   2 & 0 \\
   1 & 1
   \end{bmatrix}.
   Compute A + B, A^2 + AB + BA + B^2, and B^{-1} A^{-1}. You may save some time by thinking before plunging into the computations.

7. For what real number c will the matrix D have no inverse? Explain your answer.
   \[
   D = \begin{bmatrix}
   3 & 15 \\
   4 & c
   \end{bmatrix}
   \]

8. Let \( P = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2x2}(\mathbb{R}) \mid a d \neq b c \right\} \)

   *Fact:* The inverse of a diagonal matrix belonging to P can be found simply by reciprocating the diagonal elements of the matrix.
(a) Determine \( \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}^{-1} \)

(b) Suppose \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in P \) and \( \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} 1/a & b \\ c & 1/d \end{bmatrix} \)

In general, is \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) a diagonal matrix? If yes, explain why; if no, give the most general form of such a matrix \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \).

Section 5.5

9. (a) Let \( A \) and \( B \) be \( n \times n \) matrices. Expand \( (A + B)^2 \).
(b) Is \( (A + B)^2 \) ever equal to \( A^2 + 2AB + B^2 \)? Explain.

10. Solve the following matrix equation for \( X \). Be careful to explain under which conditions each step is possible.
\[ AX + C = BX \]

Section 5.6

11. Prove or disprove: \( A^{-1} = A \) and \( B^{-1} = B \) \( \implies \) \( (AB)^{-1} = AB \).

12. The following is true for all real numbers \( a \) and \( b \): \( a \cdot b = 0 \) if and only if \( a = 0 \) or \( b = 0 \). Is any part of this statement true for \( n \times n \) matrices \( A \) and \( B \)? Explain. Give an example and proof.

13. Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) where \( a, b, c, d \in \mathbb{R} \). Show that the matrices of the form \( A = \pm \begin{bmatrix} 1 & 0 \\ c & -1 \end{bmatrix} \) and \( A = \pm \begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix} \) are also solutions to the equation \( A^2 = I \), confirming that a quadratic matrix equation can have an infinite number of solutions. Are there any others?
CHAPTER 5 (solutions)

Sections 5.1–5.3

1. For parts c, d and i of this exercise, only a verification is needed. Here, we supply the result that will appear on both sides of the equality.

(a) \( AB = \begin{bmatrix} -3 & 6 \\ 9 & -13 \end{bmatrix} \) \( BA = \begin{bmatrix} 2 & 3 \\ -7 & -18 \end{bmatrix} \)

(b) \( \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} \)

(c) \( \begin{bmatrix} 3 \\ 0 \\ 15 \\ -6 \end{bmatrix} \) \( \begin{bmatrix} 18 & -15 & 15 \\ -39 & 35 & -35 \end{bmatrix} \)

(d) \( \begin{bmatrix} -12 \\ -21 \\ 7 \\ -6 \end{bmatrix} \)

(e) \( \begin{bmatrix} 5 \\ -5 \end{bmatrix} \)

(f) \( B + 0 = B \)

(g) \( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \)

(h) \( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \)

(i) \( \begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix} \)

3. \( \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1/3 \end{bmatrix} \)

5. \( A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 27 \end{bmatrix} \)

\( A^{15} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 32768 & 0 \\ 0 & 0 & 14348907 \end{bmatrix} \)

7. (a) \( Ax = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} \) equals \( \begin{bmatrix} 3 \\ 1 \end{bmatrix} \) if and only if both of the equalities \( 2x_1 + x_2 = 3 \) and \( x_1 - x_2 = 1 \) are true.

(i) \( A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \)

\( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \)

\( B = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \)

(ii) \( A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \end{bmatrix} \)

\( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \)

\( B = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \)

(iii) \( A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} \)

\( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \)

\( B = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} \)

Section 5.4

1. (a) \( \begin{bmatrix} -1/5 & 3/5 \\ 2/5 & -1/5 \end{bmatrix} \)

(b) No inverse exists. (c) \( \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \)
3. (a) Let $A$ and $B$ be $n$ by $n$ invertible matrices. Prove $(AB)^{-1} = B^{-1}A^{-1}$.

Proof: 

$$(B^{-1}A^{-1})(AB) = (B^{-1})(A^{-1}(AB))$$

$$(B^{-1})(B) = I$$

Similarly, $(AB)(B^{-1}A^{-1}) = I$.

So by Theorem 5.4.1, $B^{-1}A^{-1}$ is the only inverse of $AB$. If we tried to invert $AB$ with $A^{-1}B^{-1}$, we would be unsuccessful since we cannot rearrange the order of the matrices.

5. (b) $1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$. Now solve for $\det(A^{-1})$.

7. **Basis** ($n = 1$): $\det(A^1) = \det(A) = (\det A)^1$.

**Induction**: Assume $\det(A^n) = (\det A)^n$ for some $n \geq 1$.

$$\begin{align*}
\det(A^{n+1}) &= \det(A^n A) \\
&= \det(A^n)\det(A) \\
&= (\det A)^n\det(A) \\
&= (\det A)^{n+1}
\end{align*}$$

9. (a) Assume $A = BDB^{-1}$

**Basis** ($m = 1$): $A^1 = E D^1 B^{-1}$ is given.

**Induction**: Assume that for some positive integer $m$, $A^m = BD^m B^{-1}$

$$A^{m+1} = A^m A$$

$$= (BD^m B^{-1})(BDB^{-1})$$

by the induction hypothesis

$$= BD^m DB^{-1}$$

by associativity, definition of inverse

$$= BD^{m+1} B^{-1}$$

$$A^{10} = BD^{10} B^{-1} = \begin{bmatrix}
-9260 & 15345 \\
-6138 & 10231
\end{bmatrix}$$

Section 5.5

1. (1) Let $A$ and $B$ be $m$ by $n$ matrices. Then $A + B = B + A$.

(2) Let $A$, $B$, and $C$ be $m$ by $n$ matrices. Then $A + (B + C) = (A + B) + C$. 

(3) Let $A$ and $B$ be $m$ by $n$ matrices, and let $c \in \mathbb{R}$.
Then $c(A + B) = cA + cB$.

(4) Let $A$ be an $m$ by $n$ matrix, and let $c_1, c_2 \in \mathbb{R}$.
Then $(c_1 + c_2)A = c_1A + c_2A$.

(5) Let $A$ be an $m$ by $n$ matrix, and let $c_1, c_2 \in \mathbb{R}$.
Then $c_1 (c_2A) = (c_1 c_2)A$.

(6) Let $0$ be the zero matrix of size $m$ by $n$, and let $A$ be a matrix of size $n$ by $r$. Then
$0A = 0 = \text{the m by r zero matrix}$.

(7) Let $A$ be an $m$ by $n$ matrix, and $0$ = the number zero.
Then $0A = 0 = \text{the m by n zero matrix}$.

(8) Let $A$ be an $m$ by $n$ matrix, and let $0$ be the $m$ by $n$ zero matrix.
Then $A + 0 = A$.

(9) Let $A$ be an $m$ by $m$ matrix. Then $A + (-1)A = 0$, where 0 is the $m$ by $n$ zero matrix.

(10) Let $A$, $B$, and $C$ be $m$ by $n$, $n$ by $r$, and $n$ by $r$ matrices respectively.
Then $A(B + C) = AB + AC$.

(11) Let $A$, $B$, and $C$ be $m$ by $n$, $r$ by $m$, and $r$ by $m$ matrices respectively. Then
$(B + C)A = BA + CA$.

(12) Let $A$, $B$, and $C$ be $m$ by $n$, $n$ by $r$, and $r$ by $p$ matrices respectively. Then
$A(BC) = (AB)C$.

(13) Let $A$ be an $m$ by $n$ matrix, $I_m$ the $m$ by $m$ identity matrix, and $I_n$ the $n$ by $n$ identity matrix. Then $I_mA = A = I_nA$.

(14) Let $A$ be an $n$ by $n$ matrix. Then if $A^{-1}$ exists, $(A^{-1})^{-1} = A$.

(15) Let $A$ and $B$ be an $n$ by $n$ matrix. Then if $A^{-1}$ and $B^{-1}$ exist,
$(AB)^{-1} = B^{-1} A^{-1}$

3. (a) $AB + AC = \begin{bmatrix} 21 & 5 & 22 \\ -9 & 0 & -6 \end{bmatrix}$
(c) Since $A(B+C) = AB + AC$, the answer is the same as
that of part (a)
(b) $A^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$
(d) $(A^2)^{-1} = (AA)^{-1} = (AA^{-1})^{-1} = I^{-1} = I$

Section 5.6

1. In elementary algebra (the algebra of real numbers), each of the given oddities
does not exist.
(i) $AB$ may be different from $BA$.
Not so in elementary algebra, since $ab = ba$ by the commutative law of multiplication.
There exist matrices $A$ and $B$ such that $AB = 0$, yet $A \neq Q$ and $B \neq 0$. In elementary algebra, the
only way $ab = 0$ is if either $a$ or $b$ is zero. There are no exceptions.
(iii) There exist matrices $A$, $A \neq 0$, yet $A^2 = 0$. In elementary algebra,
$a^2 = 0 \iff a = 0$. 

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(iv) There exist matrices $A^2 = A$ where $A \neq 0$ and $A \neq I$. In elementary algebra, $a^2 = a \iff a = 0$ or $1$.

(v) There exist matrices $A$ where $A^2 = I$ but $A \neq I$ and $A \neq -I$. In elementary algebra, $a^2 = 1 \iff a = 1$ or $-1$.

3. (a) $\det(A) \neq 0 \implies A^{-1}$ exists, and if you multiply the equation $A^2 = A$ on both sides by $A^{-1}$, you obtain $A = I$

(b) Not true, counterexample: $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

5. (a) $A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-2}{3} \end{bmatrix}$ $x_1 = \frac{4}{3}$, and $x_2 = \frac{1}{3}$

(b) $A^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$ $x_1 = 4$, and $x_2 = 4$

(c) $A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-2}{3} \end{bmatrix}$ $x_1 = \frac{2}{3}$, and $x_2 = \frac{-1}{3}$

(d) $A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-2}{3} \end{bmatrix}$ $x_1 = 0$, and $x_2 = 1$

(e) The matrix of coefficients for this system has a zero determinant: therefore it has no inverse. The system cannot be solved by this method. In fact, the system has no solution.

Solutions for the Supplementary Exercises

1. $\begin{bmatrix} x + y & 5 \\ -2 & x - y \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix} \implies \begin{cases} x + y = 3 \\ x - y = 4 \end{cases} \implies \begin{cases} y = -\frac{1}{2} \\ x = \frac{7}{2} \end{cases}$

3. For $n \geq 1$ let $p(n)$ be $AB^n = B^n A$

Basis: ($n = 1$): $AB^1 = B^1 A$ is true as given in the statement of the problem. Therefore, $p(1)$ is true.

Induction: Assume $n \geq 1$ and $p(n)$ is true.

$AB^{n+1} = (B^n A)B$

$= (B^n A)B$ By the induction hypothesis

$= (B^n B)A$ By $p(1)$

$= (B^n B)A$

$= B^{n+1} A$
5. \[ A^{-1}A^3 = A^2 = \begin{bmatrix} 7 & 18 \\ 6 & 19 \end{bmatrix} \]

D has no inverse if \( \text{det}(D) = 0 \).

\[ \text{det}(D) = 0 \iff 3c - 4(15) = 3c - 60 = 0 \iff c = 20. \]

9.(a) \((A+B)^2 = A^2 + AB + BA + B^2\)

\((A + B)^2 = A^2 + 2AB + B^2\) only if \(AB = BA\).

11. The implication is false. Both \[ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \] and, \[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \] are self-inverting, but their product is not.

13. Yes, matrices of the form \( A = \begin{bmatrix} a & b \\ (1-a)^2/\overline{b} & -a \end{bmatrix} \), also solve \( A^2 = 1 \).