Basic Counting Techniques—The Rule of Products

WHAT IS COMBINATORICS?

One of the first concepts our parents taught us was the "art of counting." We were taught to raise three fingers to indicate that we were three years old. The question of "how many" is a natural and frequently asked question. Combinatorics is the "art of counting." It is the study of techniques that will help us to count the number of objects in a set quickly. Counting occurs not only in highly sophisticated applications of mathematics to engineering and computer science but also in many basic applications. Like many other powerful and useful tools in mathematics, the concepts are simple; we only have to recognize when and how they can be applied. The following examples will illustrate that many questions concerned with counting involve the same process.

The Product Rule is probably one of the most versatile counting techniques. Before you look at the definition consider the following basic example:

Example 1. A snack bar serves five different sandwiches and three different beverages. How many different lunches can a person order? One way of determining the number of possible lunches is by listing or enumerating all the possibilities. One systematic way of doing this is by means of a tree (see figures 2 & 3 of the text in section 4.1.). Read the figures from left to right, and for readability label the vertex on the extreme left START. I have used arrows in the following diagram to emphasize that it should be read from left to right.

Assume the five sandwich choices are: beef, cheese, chicken ham and bologna. Label the tips of each of the above arrows by these choices. For consistency label them in the order given from the top-most "arrow" to the one at the bottom. Next we have three choices of drink. Let’s assume they are: milk, juice and coffee in that order. Draw three arrows from each of the sandwich choices and label them in order milk, juice and coffee. When you fill out the above diagram completely you should have a total of 15 possible choices of lunch. Every path that begins at the position labeled START and goes to the right can be interpreted as a choice of one of the five sandwiches followed by a choice of one of the three beverages. Note that considerable work is required to arrive at the
number fifteen this way, but we also get: more than just a number. The result is a complete list of all possible lunches. For example the top most choice is beef sandwich and milk. A listing of possible lunches a person could have is:
{(BEEF, milk),(BEEF, juice), (BEEF, coffee), ..., (BOLOGNA, coffee)}.

An alternative method of solution for this example is to make the simple observation that there are five different choices for sandwiches and three different choices for beverages, so there are $5 \cdot 3 = 15$ different lunches that can be ordered. This is the Rule of Products. NEVER underestimate the power of simple observations.

**Example 2.** Let $A = \{a, b, c, d, e\}$ and $B = \{1,2,3\}$. From Section 1.4 we know how to list the elements in $A \times B = \{(a, 1), (a, 2), (a, 3), ..,(c, 3)\}$. The reader is encouraged to imitate the above figure for this example. Since the first entry of each pair can be any one of the five elements a, b, c, d, and e, and since the second can be any one of the three numbers 1, 2, and 3, it is quite clear there are $5 \cdot 3 = 15$ different elements in $A \times B$. This we already knew from our discussions on Cartesian product.

**Example 3.** A person is to complete a true-false questionnaire consisting of ten questions. How many different ways are there to answer the questionnaire? Since each question can be answered either of two ways (true or false), and there are a total of ten questions, there are $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^{10}$ different ways of answering the questionnaire. The reader is encouraged to visualize the tree diagram of this example. We formalize the procedures developed in the previous examples with the following rule and its extension.

**RULE OF PRODUCTS**

**Rule Of Products:** If two operations must be performed, and if the first operation can always be performed $p_1$ different ways and the second operation can always be performed $p_2$ different ways, then there are $p_1p_2$ different ways that the two operations can be performed. This can be extended to any number of operations/tasks.

Note: it is important that $p_2$ does not depend on the option that is chosen in the first operation. Another way of saying this is that $p_2$ is independent of the first operation. If $p_2$ is dependent on the first operation, then the rule of products does not apply.

**Example 4.** Assume in Example 1 that coffee is not served with a beef or chicken sandwich, then by inspection of Figure 2.1.1 we see that there are only thirteen different choices for lunch. The rule of products does not apply, since the choice of beverage ($p_2$) depends on one’s choice of sandwich ($p_1$).
**Sum Rule:** If two operations must be performed, and if the first operation can always be performed \( p_1 \) different ways and the second operation can always be performed \( p_2 \) different ways, and if these operations cannot be done at the same time then there are \( p_1 + p_2 \) different ways that the two operations can be performed. This can be extended to any number of operations/tasks.

**Example 5:** A student must select a project from one of three lists provided by her teacher. The three lists contain 15, 8 and 12 projects respectively. How many possible projects are there to choose from?

**Solution:** From the first list the student can choose a project any one of 15 ways, from the second any one of 8 and from the third list any one of 12 ways. So, by the sum rule we have

\[ 15 + 8 + 12 = 35 \text{ projects to choose from.} \]

**Here are a variety of problems in no particular order. How many of them can you do just using the rule of products?**

1. A family of 5 consisting of the parents and 3 children are going to be arranged in a row by a photographer. How many ways are there to arrange the 5 members of the family (no restrictions)? If the parents are to be next to each other, how many arrangements are possible?

   (parents = 2 . 1 ways)(children = 3 . 2 . 1 ways)(4)

   The 4 comes from the parents can stand in any one of 4 places first, between each of the children, and last, that is, P- C₁-P-C₂-P-C₃-P

2. How many ways can the manager of a baseball team select a pitcher and a catcher for a game if there are 5 pitchers and 3 catchers on the team?

   ans = 15

   *Consider three-letter "words" to be formed by using the vowels a, e, i, o, and u.*

3. How many different three-letter words can be formed if repetitions are not allowed?

   ans = 60

4. How many different three-letter words can be formed if repetitions are allowed?

   ans = 125

5. How many different three-letter words without repetitions can be formed whose middle letter is \( o \)?

   ans = 12
6. How many different three-letter words without repetitions can be formed whose first letter is \( e \)?
   \[ \text{ans} = 12 \]

7. How many different words without repetitions can be formed whose letters at the ends are \( u \) and \( i \)?
   \[ \text{ans} = 6 \]

8. (a) In how many different ways can the letters of \( \text{STUDY} \) be arranged using each letter only once in each arrangement? \( \text{ans} = 120 \)
   (b) How many arrangements are if the \( S \) and \( T \) are in the first two positions?
      \[ \text{ans} = 12 \]
   (c) How many arrangements are if the \( S \) and \( T \) are next to each other? \( \text{ans} = 48 \)
      This problem is like \#1 \( (2!)(3!)(4) \)
   (b) How many arrangements are if the \( S \) and \( T \) are not next to each other?
      \[ \text{ans} = 72 \]
      Total number of ways \( (120) - \) number of ways \( S \& T \) next to each other \( (48) = \) number of ways \( S \& T \) \textbf{NOT} next to each other \( (72) \)

9. To avoid electronic detection, a ship can send coded messages to neighboring ships by displaying a sequence of signal flags having different shapes. If 12 different-shaped flags are available, how many messages can be displayed using a 4-flag sequence? \( \text{ans} = 11,880 \)

10. (a) A social security number is a sequence of nine digits. How many different social security numbers are possible? \( \text{ans} = 1,000,000,000 \)
    (b) How many are there if no digits repeat? \( \text{ans} = 3,628,800 \)
    (c) How many are there if a digit appears exactly three times in succession and no other digits repeat? \( \text{ans} = 10 \times 7(9 \times 8 \times 7 \times 6 \times 5 \times 4) \)

11. An ice cream parlor advertises that you may have your choice of five different toppings, and you may choose none, one, two, three, four, or all five toppings. How many choices are there in all?
    Method 1. For each topping you have 2 choices accept the topping or not.
    Therefore have 2^5 choices.
    Method 2. (This method is covered in section 4.3) Another point of view.
    We want the number of different ways to select 0,1,2,3,4, or 5 elements from a total of 5 possibilities, that is:
    \[
    \binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 2^5
    \]
14. A basketball squad has 12 players consisting of 3 centers, 5 forwards, and 4 guards. How many ways can the coach field a team having 1 center, 2 forwards, and 2 guards?

\[
\text{ans} = 180
\]

Now read the text and try the assignment. Good luck.

**Pigeonhole Principle**

Read section 4.2. Here are some additional examples to help you understand this counting technique.

1. Explain the following; If a six sided die is cast 7 times at least one of the sides will come up twice.

2. Six friends discover the have a total of $21.61 with them. Show that one or more of them must have $3.61.

   Solution: Let \( N = 2161 \) and \( k = 6 \) then
   \[
   \left[ \frac{2161}{6} \right] = 361.1666
   \]

3. Seven colors are used to paint fifty cars. Show that at least eight cars have the same color. (Or ask How many cars have the same color?)

   Solution: Let \( N= 50 \) and \( k = 7 \) then
   \[
   \left[ \frac{50}{7} \right] = 8
   \]

4. How many friends must you have to ensure that at least five of them will have birthdays in the same month?

   Solution: Find \( N \) such that \( \left[ \frac{N}{12} \right] = 5 \). Thus \( N \) must be at least 49. Why?