Abstract  We describe a trivial solution to the Minkowski problem for polygons in the Euclidean plane.

Given a collection of unit normals $u_i$ and corresponding edge lengths $\alpha_i > 0$, one might ask whether a convex polygon $P$ exists whose boundary is described by the given normals and lengths. This question is known as the Minkowski Problem. While the Minkowski Problem (actually a theorem) is deep and non-trivial in higher dimensions (see, for example, [BF48, Sch93]), it is almost trivial in the planar case: By rotating the normals counterclockwise by $90^\circ$ and lengthening each by its given edge length, we transform the normals into actual edges. Assuming we have listed the normals in counterclockwise order, we can just lay these oriented edges end-to-end and construct the polygonal closed curve that describes the desired polygon in a unique way up to translation; i.e. depending only on where we set down the pen to draw the first edge.

The details, which require some bookkeeping, are described as follows.

**Theorem 1.1 (Minkowski Existence Theorem)** Suppose $u_1, u_2, \ldots, u_k \in \mathbb{R}^2$ are unit vectors that span $\mathbb{R}^2$, and suppose that $\alpha_1, \alpha_2, \ldots, \alpha_k > 0$. There exists a polygon $P \in \mathcal{P}^2$ having edge unit normals $u_1, u_2, \ldots, u_k$, and corresponding edge lengths $\alpha_1, \alpha_2, \ldots, \alpha_k$, if and only if

$$\alpha_1 u_1 + \cdots + \alpha_k u_k = 0.$$  \hspace{1cm} (1)

Moreover, such a polygon $P$ is unique up to translation.

**Proof:** Suppose a polygon $P$ has boundary data given by the normals $u_i$ and edge-lengths $\alpha_i$. Let $\phi$ denote the counter-clockwise rotation of $\mathbb{R}^2$ by the angle $\pi/2$. For each $i$ let $v_i = \phi(\alpha_i u_i)$. Then each $v_i$ is congruent to the $i$-th edge of the polygon $P$. Since the boundary of a convex polygon is a simple closed curve, we have

$$v_1 + \cdots + v_k = 0.$$

On applying $\phi^{-1}$ to this identity, we obtain (1).

Conversely, suppose that a family of unit vectors $u_i$ and positive real numbers $\alpha_i$ satisfy (1), where the vectors $u_i$ span $\mathbb{R}^2$. As above, let $v_i = \phi(\alpha_i u_i)$ for each $i$. Assume also that the vectors $u_i$ (and therefore, the vectors $v_i$) are indexed in counterclockwise order...
around the circle. We will construct a polygon $P$ having boundary data given by the normals $u_i$ and edge-lengths $\alpha_i$. The condition (1) implies that $v_1 + \cdots + v_k = 0$.

Denote
\[
\begin{align*}
x_1 &= v_1 \\
x_2 &= v_1 + v_2 \\
&\vdots \\
x_k &= v_1 + \cdots + v_k = o
\end{align*}
\]
and let $P$ denote the convex hull of the points $x_1, x_2, \ldots, x_k$ (where $x_k = o$, the origin). We will show that each $x_i$ is an extreme point of $P$. It will then follow that the $x_i$ are the vertices of $P$, so that the edges of $P$ are congruent to the vectors $v_i$, as required.

To show that each $x_i$ is an extreme point, it is sufficient to consider the case of $x_k = o$. Moreover, since convex dependence relations are invariant under rigid motions, we may assume without loss of generality that $v_1$ points along the positive $x$-axis. Let $j = (0, 1)$. Note that $x_1 \cdot j = v_1 \cdot j = 0$, and that if $v_s \cdot j < 0$ and $s \leq t \leq k$ then $v_t \cdot j < 0$, since the $v_i$ are ordered in counter-clockwise order. Moreover, since $\sum v_i = o$ and since the $v_i$ span $\mathbb{R}^2$, we must have $v_2 \cdot j > 0$.

Now suppose that $o$ is not an extreme point of $P$. In this case, $o = a_1 x_1 + \cdots + a_k x_k$, where each $a_i > 0$ and $a_1 + \cdots + a_k = 1$. Note that
\[
0 = o \cdot j = \sum a_i (x_i \cdot j).  \tag{2}
\]
Since $x_2 \cdot j > 0$, it follows from (2) that some $x_s \cdot j < 0$. Because $v_i \cdot j < 0$ for all $i \geq s$, we have
\[
0 = o \cdot j = x_k \cdot j = \left( x_s + \sum_{i > s} v_i \right) \cdot j < 0,
\]
a contradiction. It follows that $o$ (and similarly each other $x_i$) must be an extreme point of $P$. It also follows that $x_s \cdot j \geq 0$ for all $s$, so that $v_1$ (and similarly each other $v_i$) must be parallel to edges of $P$.

Since we have given an explicit reconstruction of the boundary of $P$ from the normals $u_i$ and edge lengths $\alpha_i$, starting from a base point, in this case the origin $o$, it also follows that such a polygon $P$ is unique up to choice of base point, that is, up to translation. ■

References
