1. A fair (6-sided) die is tossed 2 times. Let $M$ denote the maximum of the two values tossed.

(a) Compute $E(M)$.

**Solution:** To begin, let’s find the pdf $f_M$. Consideration of the six possible cases reveals that

$$f_M(1) = \frac{1}{36}, \quad f_M(2) = \frac{3}{36}, \quad f_M(3) = \frac{5}{36}, \quad f_M(4) = \frac{7}{36}, \quad f_M(5) = \frac{9}{36}, \quad f_M(6) = \frac{11}{36},$$

so that

$$E(M) = 1 \cdot \frac{1}{36} + 2 \cdot \frac{3}{36} + 3 \cdot \frac{5}{36} + 4 \cdot \frac{7}{36} + 5 \cdot \frac{9}{36} + 6 \cdot \frac{11}{36} = \frac{161}{36} \approx 4.4722$$

(b) Compute $\text{Var}(M)$.

**Solution:** Using the results above, we have

$$E(M^2) = 1 \cdot \frac{1}{36} + 4 \cdot \frac{3}{36} + 9 \cdot \frac{5}{36} + 16 \cdot \frac{7}{36} + 25 \cdot \frac{9}{36} + 36 \cdot \frac{11}{36} = \frac{791}{36},$$

so that

$$\text{Var}(M) = E(M^2) - E(M)^2 = \frac{791}{36} - \left(\frac{161}{36}\right)^2 = \frac{2555}{1296} \approx 1.9715$$
2. Let $X$ and $Y$ be two continuous random variables defined over the unit square in $\mathbb{R}^2$; that is, the region
$$[0, 1] \times [0, 1] = \{(x, y) \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}.$$ Suppose these random variables have joint density given by
$$f_{X,Y}(x,y) = \frac{4x + 6y}{5}.$$

(a) Compute $E(XY)$.

**Solution:** Note that, since $Y$ and $X$ are not independent, it is possible (even likely) that $E(XY) \neq E(X)E(Y)$, so that would be the wrong approach here. Instead, compute
$$E(XY) = \int_0^1 \int_0^1 xyf_{X,Y}(x,y) \, dy \, dx = \int_0^1 \int_0^1 xy \cdot \frac{4x + 6y}{5} \, dy \, dx$$
$$= \frac{1}{5} \int_0^1 \int_0^1 4x^2y + 6xy^2 \, dy \, dx = \frac{1}{5} \int_0^1 \left[ 2x^3y^2 + 2xy^3 \right]_y=0^1 dx$$
$$= \frac{1}{5} \int_0^1 2x^2 + 2x \, dx = \frac{1}{5} \left[ \frac{2x^3}{3} + x^2 \right]_0^1$$
$$= \frac{1}{5} \left[ \frac{2}{3} + 1 \right] = \frac{1}{5} \cdot \frac{5}{3} = \frac{1}{3}$$

(b) Compute $\text{Var}(Y)$.

**Solution:** In order to compute $\text{Var}(Y)$, let us first compute $E(Y)$ and $E(Y^2)$. We have
$$E(Y) = \int_0^1 \int_0^1 yf_{X,Y}(x,y) \, dy \, dx = \int_0^1 \int_0^1 y \cdot \frac{4x + 6y}{5} \, dy \, dx = \frac{1}{5} \int_0^1 \int_0^1 4xy + 6y^2 \, dy \, dx$$
$$= \frac{1}{5} \int_0^1 \left[ 2xy^2 + 2y^3 \right]_y=0^1 dx = \frac{1}{5} \int_0^1 2x + 2x \, dx = \frac{1}{5} \left[ x^2 + 2x \right]_0^1 = \frac{1}{5} \left[ 1 + 2 \right] = \frac{3}{5}$$

and
$$E(Y^2) = \int_0^1 \int_0^1 y^2f_{X,Y}(x,y) \, dy \, dx = \int_0^1 \int_0^1 y^2 \cdot \frac{4x + 6y}{5} \, dy \, dx = \frac{1}{5} \int_0^1 \int_0^1 4xy^2 + 6y^3 \, dy \, dx$$
$$= \frac{1}{5} \int_0^1 \left[ \frac{4x^3}{3} + \frac{3y^4}{2} \right]_y=0^1 dx = \frac{1}{5} \int_0^1 \frac{4x^3}{3} + \frac{3}{2} \, dx$$
$$= \frac{1}{5} \left[ \frac{2x^4}{3} + \frac{3x}{2} \right]_0^1 = \frac{1}{5} \left[ \frac{2}{3} + \frac{3}{2} \right] = \frac{1}{5} \left[ \frac{4 + 9}{6} \right] = \frac{13}{30}$$

so that
$$\text{Var}(Y) = E([Y]^2) - E(Y)^2 = \frac{13}{30} - \frac{9}{25} = \frac{65 - 54}{150} = \frac{11}{150}$$
3. In a certain stream of electronic data, errors occur at an average rate of twice every 10 seconds. What is the probability that more than 3 errors will occur during the next half-minute?

**Solution:** Use a Poisson model. If we measure time in seconds, then the error rate $\lambda$ is

$$\frac{2}{10} = 0.2.$$

Let $X$ be the number of errors during the next 30 seconds. Then

$$P(X = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!},$$

where $\lambda = 0.2$, and the time interval is $t = 30$, so that

$$P(X = k) = e^{-(0.2 \cdot 30)} \frac{(0.2 \cdot 30)^k}{k!} = e^{-6} \frac{6^k}{k!}.$$

The problem is asking for $P(X > 3)$. In order to avoid computing an infinite sum, note that $P(X > 3) = 1 - P(X \leq 3)$, so that

$$P(X > 3) = 1 - P(X \leq 3)$$

$$= 1 - \left[ P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) \right]$$

$$= 1 - e^{-6} \left[ \frac{6^0}{0!} + \frac{6^1}{1!} + \frac{6^2}{2!} + \frac{6^3}{3!} \right]$$

$$= 1 - e^{-6} [1 + 6 + 18 + 36]$$

$$= 1 - 61e^{-6} \approx 0.8488$$
4. Suppose that $X$ is a Poisson random variable such that

$$P(X = 1) = 0.13044 \quad \text{and} \quad P(X = 2) = 0.2087.$$ 

What is $E(X)$?

**Solution:** Suppose that $E(X) = \lambda$. Since $X$ is Poisson, we have

$$0.13044 = P(X = 1) = e^{-\lambda} \lambda,$$

while

$$0.2087 = P(X = 2) = e^{-\lambda} \frac{\lambda^2}{2!},$$

so that

$$\frac{0.2087}{0.13044} = \frac{P(X = 2)}{P(X = 1)} = \frac{e^{-\lambda} \frac{\lambda^2}{2!}}{e^{-\lambda} \lambda} = \frac{\lambda}{2},$$

so that

$$\lambda = \frac{2 \cdot 0.2087}{0.13044} = 3.2$$
5. Suppose that $U$ is a uniform random variable on the interval $[a, b]$, such that

$$E(X) = 8 \quad \text{and} \quad \text{Var}(X) = \frac{25}{3}.$$ 

Find the values of $a$ and $b$.

**Solution:** Since $U$ is uniform on $[a, b]$, we have

$$E(U) = \frac{a + b}{2} \quad \text{and} \quad \text{Var}(U) = \frac{(b - a)^2}{12},$$

so that

$$\frac{a + b}{2} = 8 \quad \text{and} \quad \frac{(b - a)^2}{12} = \frac{25}{3}.$$ 

This implies that

$$a + b = 8 \quad \text{and} \quad b - a = 10,$$

so that

$$a = 3 \quad \text{and} \quad b = 13.$$