A. Chapter 8, problem 2.

We can put these arrays into one-to-one correspondence with acceptable sequences of +1’s and −1’s. Given such an array, define \( a_k \) (for all \( k \) between 1 and \( 2n \)) to be +1 if \( k \) appears in the first row and −1 if \( k \) appears in the second row. E.g., the array

\[
\begin{bmatrix}
1 & 3 & 4 & 6 \\
2 & 5 & 7 & 8
\end{bmatrix}
\]

corresponds to the sequence +1, −1, +1, +1, −1, +1, +1, −1, −1. This sequence must contain \( n \) +1’s and \( n \) −1’s, since the array contains \( n \) entries in its first row and \( n \) entries in its second row. Furthermore, the sequence must be acceptable (in the sense defined on page 268). For, suppose the partial sum \( a_1 + \ldots + a_k \) were negative. Let \( s \) (respectively, \( t \)) be the number of positive (respectively, negative) terms in the partial sequence \( a_1, \ldots, a_k \). Then the \( t \)th column of the array contains a \( k \) in its second row and an entry larger than \( k \) in its first row, contradicting the condition stated in the problem. Hence the sequence \( a_1, \ldots, a_{2n} \) is acceptable. Conversely, given any acceptable sequence of \( n \) +1’s and \( n \) −1’s, we can create a valid array by listing in the first row (in increasing order) all the \( k \)’s for which \( a_k = +1 \) and listing in the second row (in increasing order) all the \( k \)’s for which \( a_k = -1 \).

So the number of arrays satisfying the stated conditions equals the number of acceptable sequences, which is \( C_n \cdot \left( \frac{1}{n+1} \binom{2n}{n} \right) \) and \( \binom{2n}{n} \frac{1}{n(n+1)!} \) are other acceptable answers.)

B. Find (and prove) a formula for the number of integer sequences \( a_1, a_2, \ldots, a_n \) with \( 1 \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq n \) and \( a_k \geq k \) for all \( k \).

Guessing the formula: We systematically list all the possibilities, for some small values of \( n \). E.g., when \( n = 1 \), the only possibility is \( a_1 = 1 \). When \( n = 2 \), there are two possibilities: \( (a_1, a_2) = (1,2) \) and \( (a_1, a_2) = (2,2) \). For \( n > 2 \), we need to be systematic. Here’s an example of how to be systematic when \( n = 3 \):
$a_1$ can be 1, 2, or 3.

If $a_1 = 1$, $a_2$ can be 2 or 3.

If $a_1 = 1$ and $a_2 = 2$, then $a_3$ can only be 3.

If $a_1 = 1$ and $a_2 = 3$, then $a_3$ can only be 3.

If $a_1 = 2$, $a_2$ can be 2 or 3.

If $a_1 = 2$ and $a_2 = 2$, then $a_3$ can only be 3.

If $a_1 = 2$ and $a_2 = 3$, then $a_3$ can only be 3.

If $a_1 = 3$, $a_2$ can only be 3.

If $a_1 = 3$ and $a_2 = 3$, then $a_3$ can only be 3.

So the possibilities for $(a_1, a_2, a_3)$ are (1,2,3), (1,3,3), (2,2,3), (2,3,3), and (3,3,3) (five possibilities all told). Similarly, for $n = 4$, one can check that the possibilities for $(a_1, a_2, a_3, a_4)$ are (1,2,3,4), (1,2,4,4), (1,3,3,4), (1,3,4,4), (1,4,4,4), (2,2,3,4), (2,2,4,4), (2,3,3,4), (2,3,4,4), (2,4,4,4), (3,3,3,4), (3,3,4,4), (3,4,4,4), and (4,4,4,4) (fourteen possibilities all told). So, the number of possibilities goes 1, 2, 5, 14, . . . as $n$ goes from 1 to infinity, and it’s natural to conjecture that the answer is $C_n$.

**Proving the formula:** We can put these sequences into one-to-one correspondence with the paths discussed in the Example on page 271 that stay above diagonal. For example, in the case $n = 4$ (shown on page 271), consider the path $P$ that goes north, north, east, north, east, north, east, east. If we look under this path (more precisely, if we look in the region bounded between the path $P$ and the “reference path” $P_0$ that goes east, east, east, east, north, north, north, north), we see a stack of 2 squares, and to the right of that, a stack of 3 squares, and to the right of that, a stack of 4 squares, and to the right of that, a stack of 4 squares. This gives us the sequence 2, 3, 4, 4.

More generally, if we have a path $P$ consisting of $n$ eastward steps and $n$ northward steps, we can look at the height of the $k$th eastward step in the picture (which is equal to the number of northward steps that precede the $k$th eastward steps), and call this $a_k$. Then we have $a_1 \leq a_2 \leq \ldots \leq a_n$, and moreover, the fact that the path $P$ never crosses below the diagonal implies that $a_k \geq k$ for $k = 1, 2, \ldots, n$. 


Conversely, every sequence $1 \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq n$ with $a_1 \geq 1$, $a_2 \geq 2$, \ldots, $a_n \geq n$ gives a path $P$ that goes from Home to Office without ever crossing below the diagonal line that joins Home to Office.

Therefore, the number of sequences $a_1, \ldots, a_n$ satisfying the stated conditions equals the number of paths from Home to Office that never go below the diagonal, which we know is equal to $C_n$.

C. Repeat the Example from the middle of page 270, but this time assume that the cash register starts with a single 50 cent piece (rather than starting empty). We still assume that there are $2n$ people in line to get into the theatre, that admission costs 50 cents, that $n$ of the people in line have a 50 cent piece and $n$ of them have a 1 dollar bill. In how many ways can the people line up so that whenever a person with a 1 dollar bill buys a ticket, the box office has a 50 cent piece in order to make change?

For $1 \leq i \leq 2n$, let $a_i$ be $+1$ if the $i$th person in line has a 50 cent piece and $-1$ otherwise, so that $\sum_{i=1}^{2n} a_i = 0$. Then the cashier will always be able to make change right away provided that no partial sum of the $a_i$ sequence is less than $-1$. Call a sequence of $n +1$’s and $n -1$’s inadmissible if one of its partial sums is less than $-1$, and admissible otherwise. We can show that the number of inadmissible sequences is $\binom{2n}{n-2}$. For, suppose the sequence $a_1, \ldots, a_{2n}$ is inadmissible. Then there must exist a smallest $k$ such that the partial sum $a_1 + a_2 + \ldots + a_{k-1} = -1$ and $a_k = -1$. We now reverse the signs of each of the first $k$ terms. The resulting sequence $a'_1, a'_2, \ldots, a'_{2n}$ is a sequence of $(n +2)$ $+1$’s and $(n -2)$ $-1$’s. The process is reversible, just as in the Example. So there are as many inadmissible sequences as there sequences of $(n +2)$ $+1$’s and $(n -2)$ $-1$’s. That is, the number of inadmissible sequences is $\binom{2n}{n+2}$. Since the total number of sequences of $n +1$’s and $n -1$’s is $\binom{2n}{n}$, the number of admissible sequences is $\binom{2n}{n} - \binom{2n}{n+2}$.

Alternative solution: Define “admissible” sequences as in the previous paragraph. If we are given an admissible sequence, so that all the partial sums are $\geq -1$, sticking an extra $+1$ at the front and an extra
−1 at the end gives rise to an acceptable sequence (since all the partial sums are now \( \geq 0 \), and the sum of all the terms is still 0). Conversely, every acceptable sequence must start with a +1 and end with a −1, and removing these two terms gives rise to an admissible sequence. Thus there is a one-to-one correspondence between the admissable sequences containing \( n +1 \)'s and \( n −1 \)'s and the acceptable sequences containing \((n+1)+1\)'s and \((n+1)−1\)'s. Thus the number of admissible sequences is \( C_{n+1} \), the \( n +1 \)st Catalan number. (It can be checked that \((\binom{2n}{n}) - (\binom{2n}{n+2}) = \frac{1}{n+2}(\binom{2n+2}{n+1})\), so the two methods have given the same answer.)

D. Chapter 8, problem 7. Express both \( h_n \) and \( \sum_{k=0}^{n} h_k \) as polynomials in \( n \) in the ordinary way.

The difference table for \( h_n \) is

\[
\begin{array}{cccc}
1 & -1 & 3 & 10 \\
-2 & 4 & 7 & \\
6 & 3 & \\
-3 & \\
\end{array}
\]

So we have \( h_n = 1\binom{n}{0} - 2\binom{n}{1} + 6\binom{n}{2} - 3\binom{n}{3} = -\frac{1}{2}n^3 + \frac{9}{2}n^2 - 6n + 1 \) and
\[
\sum_{k=0}^{n} h_k = 1\binom{n+1}{1} - 2\binom{n+1}{2} + 6\binom{n+1}{3} - 3\binom{n+1}{4} = -\frac{1}{8}n^4 + \frac{5}{8}n^3 - \frac{7}{8}n^2 - \frac{5}{8}n + 1.
\]

E. Chapter 8, problem 8. Express \( \sum_{k=1}^{n} k^5 \) as a polynomial in \( n \) in the ordinary way.

Note that \( \sum_{k=1}^{n} k^5 = \sum_{k=0}^{n} k^5 \).

The difference table for fifth powers is

\[
\begin{array}{ccccccc}
0 & 1 & 32 & 243 & 1024 & 3125 \\
1 & 31 & 211 & 781 & 2101 \\
30 & 180 & 570 & 1320 \\
150 & 390 & 750 \\
240 & 360 \\
120 & \\
\end{array}
\]

so the polynomial for adding fifth powers is \( 0\binom{n+1}{1} + 1\binom{n+1}{2} + 30\binom{n+1}{3} + 150\binom{n+1}{4} + 240\binom{n+1}{5} + 120\binom{n+1}{6} = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2 \).