1.4 MODELING AND ENERGY METHODS

Modeling is the art or process of writing down an equation, or system of equations, to describe the motion of a physical device. For example, equation (1.2) was obtained by modeling the spring–mass system of Figure 1.4. By summing the forces acting on the mass in the x direction and employing the experimental evidence of the mathematical model of the force in a spring given by Figure 1.3, equation (1.2) can be obtained. The success of this model is determined by how well the solution of equation (1.2) predicts the observed behavior of the system. This comparison between the vibration response of a device and the response predicted by the analytical model is discussed in Section 1.6. The majority of this book is devoted to the analysis of vibration models. However, two methods of modeling—Newton’s law and energy methods—are presented in this section. More comprehensive treatments of modeling can be found in Doebelein (1980), Shames (1980, 1989), and Cannon (1967), for example.

The force summation method is used in the previous sections and should be familiar to the reader from introductory dynamics (see, e.g., Shames, 1980). Newton’s law of motion (called Newton’s second law) states that the rate of change of the absolute momentum of the mass center is proportional to the net applied force vector and acts in a direction of the net force. For systems with constant mass (such as those considered here) moving in only one direction, the rate of change of momentum becomes the scalar relation

\[
\frac{d}{dt} (mx) = \dot{m}x
\]

which is often called the inertia force. The physical device of interest is examined by noting the forces acting on the center of mass. The forces are then summed (as vectors) to produce a dynamic equation following Newton’s law. For motion in the x direction only, this becomes the scalar equation

\[
\sum_i f_{xi} = m\ddot{x}
\]

(1.49)

where \( f_{xi} \) denotes the \( i \)th force acting on the mass \( m \) in the \( x \) direction and the summation is over the number of such forces. In the first three chapters, only single-degree-of-freedom systems moving in one direction are considered; thus Newton’s law takes on a scalar nature. In more practical problems with many degrees of freedom, energy considerations can be combined with the concepts of virtual work to produce Lagrange’s equations, as discussed in Section 4.7. Lagrange’s equations also provide an energy-based alternative to summing forces to derive equations of motion.

For bodies that are free to rotate about a fixed axis, the sum of the torques about the rotation axis through the center of mass of the object must equal the rate of change of angular momentum of the mass. This is expressed as

\[
\sum_i M_{th} = J\ddot{\theta}
\]

(1.50)

where \( M_{th} \) are the torques acting on the object through point 0, \( J \) is the moment of inertia (sometimes denoted \( I_0 \)) about the rotation axis, and \( \dot{\theta} \) is the angle of rotation. This is discussed in more detail in Example 1.5.1.

If the forces or torques acting on an object or mechanical part are difficult to determine, an energy approach may be more efficient. In this method the differential equation of motion is established by using the principle of energy conservation. This principle is equivalent to Newton’s law for conservative systems and states that the sum of the potential energy and kinetic energy of a particle remains constant at each instant of time throughout the particle’s motion. Integrating Newton’s law
(F = mẍ) over an increment of displacement and identifying the work done in a conservative field as the change in potential energy yields

\[
U_1 - U_2 = T_2 - T_1
\]  
(1.51)

where \(U_1\) and \(U_2\) represent the particle's potential energy at the times \(t_1\) and \(t_2\), respectively, and \(T_1\) and \(T_2\) represent the particle's kinetic energy at times \(t_1\) and \(t_2\), respectively. Equation (1.51) can be rearranged to yield

\[
T + U = \text{constant}
\]  
(1.52)

where \(T\) and \(U\) denote the total kinetic and potential energy, respectively.

For periodic motion, if \(t_1\) is chosen to be the time at which the moving mass passes through its static equilibrium position, \(U_1\) can be set to zero at that time, and if \(t_2\) is chosen as the time at which the mass undergoes its maximum displacement so that its velocity is zero \((T_2 = 0)\), equation (1.51) yields

\[
T_1 = U_2
\]  
(1.53)

Since the reference potential energy \(U_1\) is zero, \(U_2\) in equation (1.53) is the maximum value of potential energy in the system. Because the energy in this system is conserved, \(T_2\) must also be a maximum value so that equation (1.53) yields

\[
T_{\text{max}} = U_{\text{max}}
\]  
(1.54)

for conservative systems undergoing periodic motion. Since energy is a scalar quantity, using the conservation of energy yields a possibility of obtaining the equation of motion of a system without using vectors.

Equations (1.52), (1.53), and (1.54) are three statements of the conservation of energy. Each of these can be used to determine the equation of motion of a spring–mass system. As an illustration, consider the energy of the spring–mass system of Figure 1.14, hanging in a gravitational field of strength \(g\). The effect of adding the mass \(m\) to the massless spring of stiffness \(k\) is to stretch the spring from its rest position at 0 to the static equilibrium position \(\Delta\). The total potential energy of the spring–mass system is the sum of the potential energy of the spring (or strain energy; see, e.g., Shames, 1989) and the gravitational potential energy. The potential energy of the spring is given by

\[
U_{\text{spring}} = \frac{1}{2} k (\Delta + x)^2
\]  
(1.55)

Figure 1.14 (a) A spring–mass system hanging in a gravitational field. Here \(\Delta\) is the static equilibrium position and \(x\) is the displacement from equilibrium. (b) The free-body diagram for static equilibrium.
The gravitational potential energy is

\[ U_{\text{grav}} = -mgx \]  

(1.56)

where the minus sign indicates that the mass is located below the reference point \( x_0 \).

The kinetic energy of the system is

\[ T = \frac{1}{2} mx^2 \]  

(1.57)

Substituting these energy expressions into equation (1.52) yields

\[ \frac{1}{2} mx^2 - mgx + \frac{1}{2} k(\Delta + x)^2 = \text{constant} \]  

(1.58)

Differentiating this expression with respect to time yields

\[ \dot{x}(m\ddot{x} + kx) + \dot{x}(k\Delta - mg) = 0 \]  

(1.59)

Since the static force balance on the mass from Figure 1.14(b) yields the fact that \( k\Delta = mg \), equation (1.59) becomes

\[ \dot{x}(m\ddot{x} + kx) = 0 \]  

(1.60)

The velocity \( \dot{x} \) cannot be zero for all time; otherwise, \( x(t) = \) constant and no vibration would be possible. Hence equation (1.60) yields the standard equation of motion

\[ m\ddot{x} + kx = 0 \]  

(1.61)

This procedure is called the energy method of obtaining the equation of motion.

The energy method can also be used to obtain the frequency of vibration directly for conservative systems that are oscillatory. The maximum value of sine (and cosine) is 1. Hence, from equations (1.3) and (1.4), the maximum displacement is \( A \) and the maximum velocity is \( \omega_n A \). Substitution of these maximum values into the expression for \( U_{\text{max}} \) and \( T_{\text{max}} \) and using the energy equation (1.54) yields

\[ \frac{1}{2} m(\omega_n A)^2 = \frac{1}{2} kA^2 \]  

(1.62)

Solving this for \( \omega_n \) yields the standard natural frequency relation \( \omega_n = \sqrt{k/m} \).
Example 1.4.1

Figure 1.15 is a crude model of a vehicle suspension system hitting a bump. Calculate the natural frequency of oscillation using the energy method. Assume that no energy is lost during the contact.

![Figure 1.15 Simple model of an automobile suspension system. The rotation of the wheel relative to the horizontal as it hits a bump is given by $\theta$. It is assumed that the wheel rolls without slipping as the car hits the bump.](image)

**Solution** From introductory dynamics, the rotational kinetic energy of the wheel is $T_{\text{rot}} = \frac{1}{2}J \dot{\theta}^2$ where $J$ is the mass moment of inertia of the wheel and $\theta = \theta(t)$ is the angle of rotation of the wheel. This assumes that the wheel moves relative to the surface without slipping as it climbs the bump (so that no energy is lost at contact). The translational kinetic energy of the wheel is $T_I = \frac{1}{2}m \dot{x}^2$.

The rotation $\theta$ and the translation $x$ are related by $x = r\theta$. Thus $\dot{x} = r\dot{\theta}$ and $T_{\text{rot}} = \frac{1}{2}J \dot{\theta}^2/r^2$. At maximum energy $x = A$ and $\dot{x} = \omega_n A$, so that

$$T_{\text{max}} = \frac{1}{2} m \dot{x}_{\text{max}}^2 + \frac{1}{2} \frac{J}{r^2} \dot{\theta}_{\text{max}}^2 = \frac{1}{2} (m + J/r^2) \omega_n^2 A^2$$

and

$$U_{\text{max}} = \frac{1}{2} k x_{\text{max}}^2 = \frac{1}{2} k A^2$$

Using conservation of energy in the form of equation (1.54) yields $T_{\text{max}} = U_{\text{max}}$, or

$$\frac{1}{2} \left( m + \frac{J}{r^2} \right) \omega_n^2 = \frac{1}{2} k$$

Solving this last expression for $\omega_n$ yields

$$\omega_n = \sqrt{\frac{k}{m + J/r^2}}$$

the desired frequency of oscillation of the suspension system.

The denominator in the frequency expression derived in this example is called the **effective mass** because the term $(m + J/r^2)$ has the same effect on the natural frequency as does a mass of value $(m + J/r^2)$. 