Derivatives and the Shapes of Graphs

\[ f(x) = \frac{1}{3} x^3 + \frac{1}{2} x^2 - 2x \quad f'(x) = x^2 + x - 2 \quad f''(x) = 2x + 1 \]
What $f'(x)$ says about $f(x)$.

- **The sign of $f'(x)$**: Determines where the function $f(x)$ increases or decreases. Notice that the function rises to the left of $A$, falls between $A$ and $C$, then rises again to the right of $C$. This information is given in the graph of $f'(x)$. Notice that the function $f'(x)$ is positive to the left of $A'$ and the right of $C'$. Between $A'$ and $C'$ the function $f'(x)$ is negative.

- **Stationary, Maximum, and Minimum Points**: The points $A$ and $C$ where the function $f(x)$ is relatively high and low occur where $f'(x) = 0$, $A'$, and $C'$. These are called stationary points.

- **Concavity and Inflection**: At the point $B$, the function’s concavity changes from concave down to concave up. This corresponds to the point where the graph of $f'(x)$ changes from decreasing to increasing.
Increasing and Decreasing Test

1. If \( f'(x) > 0 \) on an interval, then \( f(x) \) increases on that interval.

2. If \( f'(x) < 0 \) on an interval, then \( f(x) \) decreases on that interval.

First Derivative Test

Suppose that \( c \) is a critical number of a continuous function \( f \).

- If \( f'(x) > 0 \) for \( x < c \), and \( f'(x) < 0 \) for \( x > c \), then \( c \) is a local maximum point. i.e. If \( f' \) changes from positive to negative at \( c \).

- If \( f'(x) < 0 \) for \( x < c \), and \( f'(x) > 0 \) for \( x > c \), then \( c \) is a local minimum point. i.e. If \( f' \) changes from negative to positive at \( c \).

- If \( f' \) does not change sign at \( c \), then \( f \) has no local extreme value at \( c \).
Concavity, and Inflection Points

The graph of \( f(x) \) is **concave up** at \( x = c \) if the slope function \( f'(x) \) is increasing at \( x = c \). The graph of \( f(x) \) is **concave down** at \( x = c \) if the slope function \( f'(x) \) is decreasing at \( x = c \). A point where the function changes concavity is called an **inflection point**.

Concavity Test

- If \( f''(x) > 0 \) for all \( x \) in \( I \), then the graph of \( f \) is concave upward on \( I \).
- If \( f''(x) < 0 \) for all \( x \) in \( I \), then the graph of \( f \) is concave downward on \( I \).
Second Derivative Test: Suppose that $f''$ is continuous near $c$.

- If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has a local minimum at $x = c$.
- If $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has a local maximum at $x = c$.

Note: If $f''(c) = 0$, then anything can happen. $f(x)$ may have a local minimum, a local minimum, or neither at $x = c$. 

Example: Determine intervals on which the function \( f(x) = 2x^3 - 3x^2 + 4 \) is increasing or decreasing. Find local maximum and minimum values, and then intervals on which the function is concave up or concave down and the inflection points.

Solution: First compute the first derivative:

\[
f'(x) = 6x^2 - 6x = 6x(x - 1)
\]

\( f' = 0 \) at \( x = 0 \), and 1, so \( x = 0 \), and 1 are critical numbers.

\( f' > 0 \) for \(( -\infty, 0) \cup (1, \infty)\), and \( f' < 0 \) for \( 0 < x < 1 \).

This means that there is a relative maximum value of 4 at \( x = 0 \), and a relative minimum value of 3 at \( x = 1 \).
Computing the second derivative;

\[ f''(x) = 12x - 6 = 6(2x - 1) \]

Now, the sign of \( f''(x) \) is determined. The zeros of \( f''(x) \) are given by \( 2x - 1 = 0 \) which has solution \( x = \frac{1}{2} \).

For \( x < \frac{1}{2} \) \( f''(x) < 0 \), so \( f \) is concave down there.

For \( \frac{1}{2} < x \) \( f''(x) > 0 \) so \( f \) is concave up there.
Since the function changes concavity at \( x = \frac{1}{2} \), the function has an inflection point there. Since \( f\left(\frac{1}{2}\right) = 2\left(\frac{1}{8}\right) - 3\left(\frac{1}{4}\right) + 4 = \frac{7}{2} \), the point of inflection are \( \left(\frac{1}{2}, \frac{7}{2}\right) \).
Example: Let $f(x) = \frac{x}{x^2 + 1}$. Find out everything about this function and then graph it.

First notice that $f(-x) = \frac{-x}{(-x)^2 + 1} = -f(x)$, so the function is symmetric through the origin.

Since $f'(x) = \frac{(1)(x^2 + 1) - (x)(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$, the sign and zeroes of $f'$ are determined by the sign (zeroes) of $(1 - x^2) = (1 + x)(1 - x)$.

Stationary points of $f$ are at $x = \pm 1$. $f' > 0$ for $x \in (-1, 1)$, and $f' < 0$ for $x < -1$, and $x > 1$. So, $f(-1) = -\frac{1}{2}$ is a relative minimum, and $f(1) = \frac{1}{2}$ is a relative maximum.
Taking the second derivative, \( f'(x) = \frac{(1 - x^2)}{(x^2 + 1)^2} \)

\[
f''(x) = \frac{(-2x)(x^2 + 1)^2 - (1 - x^2)[4x(x^2 + 1)]}{(x^2 + 1)^4} = \frac{(-2x)(x^2 + 1) - (1 - x^2)(4x)}{(x^2 + 1)^3}
\]

so

\[
= \frac{2x^3 - 6x}{(x^2 + 1)^3} = \frac{2x(x^2 - 3)}{(x^2 + 1)^3} = \frac{2x(x + \sqrt{3})(x - \sqrt{3})}{(x^2 + 1)^3}
\]

So \( x = 0, \pm \sqrt{3} \) are important points for \( f'' \)

So, \( f \) is concave up on \(( -\infty, -\sqrt{3})\) and \((0, \sqrt{3},)\) and concave down on \(( -\sqrt{3}, 0)\) and \((\sqrt{3}, \infty)\). There are inflection points at \((-\sqrt{3}, -\sqrt{3}/4), (\sqrt{3}, \sqrt{3}/4)\) and \((0, 0)\).
Example: Determine intervals on which the function \( f(x) = \frac{x^2}{x^2 + 1} \) is increasing or decreasing. Find local maximum and minimum values, and then intervals on which the function is concave up or concave down and the inflection points.

Solution: First note that the function is always positive and has symmetry about the \( y \)-axis. Also \( \lim_{x \to \pm \infty} f(x) = \lim_{x \to \pm \infty} \frac{x^2}{x^2 + 1} = 1 \). So there is a horizontal asymptote at \( y = 1 \), for \( x \to \pm \infty \). Now:

Compute the first derivative:

\[
f'(x) = \frac{2x(x^2 + 1) - x^2(2x)}{(x^2 + 1)^2} = \frac{2x}{(x^2 + 1)^2}
\]

\( f' = 0 \) at \( x = 0 \), so \( x = 0 \) is a critical number. Note: \( (x^2 + 1)^2 \geq 1 \).
$f' > 0$ for $x > 0$, and $f' < 0$ for $x < 0$. So $f$ is increasing for positive $x$ and decreasing for negative $x$. This means that there is a relative minimum value of 0 at $x = 0$.

Computing the second derivative;

$$f''(x) = \frac{2(x^2 + 1)^2 - 2x(2(x^2 + 1)(2x))}{(x^2 + 1)^4} = \frac{2 - 6x^2}{(x^2 + 1)^3}$$

Now, the sign of $f''(x)$ is determined by $2 - 6x^2$ alone, since the denominator is again always positive. The zeros of $f''(x)$ are given by $2 - 6x^2 = 0$ which has solutions,

$$x^2 = \frac{1}{3}, \text{ or}$$

$$x = \pm \frac{1}{\sqrt{3}}.$$
For \(-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}\) \(f''(x) > 0\), so \(f\) is concave up.

For \(-\infty < x < -\frac{1}{\sqrt{3}} \cup \frac{1}{\sqrt{3}} < x < \infty\) \(f''(x) < 0\) so \(f\) is concave down.

Since the function changes concavity at \(x = \pm \frac{1}{\sqrt{3}}\), the function has inflection points there. Since \(f\left(\pm \frac{1}{\sqrt{3}}\right) = \frac{1}{3} + \frac{1}{4} = \frac{1}{4}\), the points of inflection are \(\left(\pm \frac{1}{\sqrt{3}}, \frac{1}{4}\right)\).
Example: Determine intervals on which the function \( f(x) = x \ln x \) is increasing or decreasing. Find local maximum and minimum values, and then intervals on which the function is concave up or concave down and the inflection points.

Solution: First note that the domain of \( f \) is \( x > 0 \). Now since
\[
f'(x) = \ln x + x \cdot \frac{1}{x} = \ln x + 1,
\]
critical numbers come from \( \ln x + 1 = 0 \).

This happens for \( x = e^{-1} = \frac{1}{e} \). For \( 0 < x < e^{-1} \) \( f' < 0 \), and so the function \( f \) is decreasing there. For \( x > e^{-1} \) \( f' > 0 \) and \( f \) increases.

The first derivative test indicates that \( f(e^{-1}) = e^{-1} \ln e^{-1} = -e^{-1} \) is a relative minimum value.
Since $f''(x) = \frac{1}{x}$, the function is always concave up, and hence there are no points of inflection.
Example: Let \( f(x) = x^4 - 18x^2 + 40 \). Find intervals on which the function is increasing and decreasing; concave up and down. Locate and classify all extrema, as well as points of inflection. Using this information graph the function.

Here \( f'(x) = 4x^3 - 36x \), and \( f''(x) = 12x^2 - 36 \). We can factor the first derivative as \( f'(x) = 4x(x + 3)(x - 3) \) which has zeros at \( x = 0, \pm 3 \). and diagram its sign below.

\[ \begin{align*}
-4 & \quad -3 & \quad -2 & \quad -1 & \quad 0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 \\
\text{Sign of } f'(x) & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
- & - & - & + & + & + & + & + & + & + \\
\end{align*} \]

\( f(x) \) is increasing on the intervals \((-3, 0)\) and \((3, \infty)\) and decreasing on \((-\infty, -3)\) and \((0, 3)\).

\( x = -3, 0, 3 \) are stationary points. From the diagram \( f(\pm3) = -41 \) are relative minimum values and \( f(0) = 40 \) is a relative maximum value.
With \( f''(x) = 12x^2 - 36 \), the zeros are \( 12x^2 - 36 = 12(x^2 - 3) = 0 \), at \( x = \pm \sqrt{3} \).

Diagramming the second derivative:

Concave up on the intervals \((-\infty, -\sqrt{3})\) and \((\sqrt{3}, \infty)\)
Concave down on the interval \((-\sqrt{3}, \sqrt{3})\).
Since the concavity changes at \( x = \pm \sqrt{3} \), there are inflection points there.
Note; the actual point of inflection is \((\pm \sqrt{3}, 22)\).
Putting it all together:

Sign of $f'(x)$

Sign of $f''(x)$