Curve Sketching

1. Domain
2. Intercepts
3. Symmetry
4. Asymptotes
5. Intervals of Increase or Decrease
6. Local Maximum and Minimum Values
7. Concavity and Points of Inflection
8. Sketch the curve
Using these steps let’s analyze the function \( f(x) = \frac{1}{4}x^4 - 2x^2 + 1 \).

1) Domain: Since \( f \) is a polynomial, \(( -\infty, \infty )\) is the domain.

2) Intercepts: \( y \)-intercept is \((0, 1)\)
   
x intercepts come from solving \( \frac{1}{4}x^4 - 2x^2 + 1 = 0 \),
   
   Using \( w = x^2 \) gives \( \frac{1}{4}w^2 - 2w + 1 = 0 \ w = 4 \pm 2\sqrt{3} \),
   
   and so \( x = \pm\sqrt{4 \pm 2\sqrt{3}} \).

3) Symmetry: Since \( f(-x) = f(x) \), the function is symmetric w.r.t. \( y \)-axis.

4) Asymptotes: None (Since it’s a polynomial.)
5) **Intervals of Increase and Decrease.**

Here \( f'(x) = x^3 - 4x \), and \( f''(x) = 3x^2 - 4 \). We can factor the first derivative as \( f'(x) = x(x + 2)(x - 2) \) and diagram its sign below.

\[ \begin{array}{cccccccc}
-4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\
\hline
- & - & - & - & + & + & + & + & + \\
\end{array} \]

**Sign of \( f'(x) \)**

\( f(x) \) is increasing on the intervals \((-2, 0)\) and \((2, \infty)\) and decreasing on \((-\infty, -2)\) and \((0, 2)\). \( x = -2, 0, 2 \) are critical points.

6) **Relative Extrema:** From the diagram \( f(\pm 2) = -3 \) are relative minimum values and \( f(0) = 1 \) is a relative maximum value.
7) Concavity: Diagramming the second derivative $f''(x) = 3x^2 - 4$:

Concave up on the intervals $(-\infty, \frac{-2}{\sqrt{3}})$ and $\left(\frac{2}{\sqrt{3}}, \infty\right)$

Concave down on the interval $\left(\frac{-2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$

Since the concavity changes at $x = \pm \frac{2}{\sqrt{3}}$, there are inflection points there.

Note; the actual point of inflection is $\left(\pm \frac{2}{\sqrt{3}}, -\frac{11}{9}\right)$
Graph:

\( f'(x) \)

\( f''(x) \)
Using these steps let’s examine the function \( f(x) = \frac{x^2}{x^3 - 1} \).

1) Domain: This function is defined for \( x \neq 1 \).

2) Intercepts: \( y \)-intercept is \((0, 0)\) \( x \)-intercept is \((0, 0)\).

3) Symmetry: Since \( f(-x) \neq f(x) \) and \( f(-x) \neq -f(x) \) the function has no typical symmetry.

4) Asymptotes: \( \lim_{x \to \pm \infty} f(x) = 0 \) so, \( y = 0 \) is a horizontal asymptote,
   \( \lim_{x \to 1^+} f(x) = \infty \quad \lim_{x \to 1^-} f(x) = -\infty \) so \( x = 1 \) is a vertical asymptote.
5) Intervals of Increase and Decrease.

\[ f'(x) = \frac{2x(x^3 - 1) - x^2(3x^2)}{(x^3 - 1)^2} = \frac{-2x - x^4}{(x^3 - 1)^2} = \frac{-x(2 + x^3)}{(x^3 - 1)^2} \]

Critical Numbers are \( x = 0, 1, \frac{3}{\sqrt{2}} \)

- - - - - - - - - + + + + + + + - - - - - - - - - - - - -

\( \text{Sign of } f'(x) \)

-2 -1 0 1 2  \( x \)

So \( f \) is decreasing on \(( -\infty, -\frac{3}{\sqrt{2}} ) \cup (0,1) \cup (1,\infty) \), and increasing on \( (-\frac{3}{\sqrt{2}}, 0) \).

6) It’s pretty clear that there is a relative minimum value at \( x = -\frac{3}{\sqrt{2}} \), the value is \( -\frac{2^{2/3}}{3} \approx -0.5291 \). And a relative maximum value of 0, at \( x = 0 \)
7) Working with \( f'(x) = \frac{-2x - x^4}{(x^3 - 1)^2} \).

\[
f''(x) = \frac{(-2 - 4x^3)(x^3 - 1)^2 + (2x + x^4)[2(x^3 - 1)(3x^2)]}{(x^3 - 1)^4}
= \frac{(-2 - 4x^3)(x^3 - 1) + (2x + x^4)[2(3x^2)]}{(x^3 - 1)^3}
= \frac{2 - 2x^3 + 4x^3 - 4x^6 + 12x^3 + 6x^6}{(x^3 - 1)^3}
= \frac{2x^6 + 14x^3 + 2}{(x^3 - 1)^3} = \frac{2(x^6 + 7x^3 + 1)}{(x^3 - 1)^3}
\]

To examine concavity and points of inflection we need find solution to \( x^6 + 7x^3 + 1 = 0 \).

Using the substitution \( w = x^3 \), we’re really solving a quadratic equation \( w^2 + 7w + 1 = 0 \), which has solutions:
\[
w = \frac{-7 \pm \sqrt{49 - 4}}{2} = \frac{-7 \pm 3\sqrt{5}}{2} \approx -0.145898, \ -6.854102
\]

Now the \( x \) values are \( x = \frac{3}{2} \sqrt{w} = c_1, c_2 = \frac{3}{2} \sqrt{\frac{-7\pm3\sqrt{5}}{2}} \approx -0.52644 \ -1.89955. \)

Sign of \( f^{''}(x) \)

So the function \( f' \) is concave up on \((c_1, c_2) \cup (1, \infty)\), and concave down on \((-\infty, c_1) \cup (c_2, 1)\).

Points of inflections are \( \left( \frac{3}{2} \sqrt{\frac{-7\pm3\sqrt{5}}{2}}, \frac{3}{2} (-7 \pm 3\sqrt{5})^{\frac{7}{5}} \right) \)
Example: Let $f(x) = xe^{-x}$. Find out everything about this function and then graph it.

1) Domain: $(-\infty, \infty)$

2) Intercepts: $y$-intercept is $(0, 0)$

$x$ intercepts come from solving $xe^{-x} = 0$, so $x = 0$.

3) Symmetry: No Symmetry

4) Asymptotes: Since $\lim_{x \to \infty} f(x) = 0$, there is a horizontal asymptote as $x$ approaches infinity.

5,6) Intervals of Increase and Decrease. Rel. Extrema

Since $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$, $x = 1$ is a stationary point. Also, $f'(x) > 0$ for $x < 1$, and $f'(x) < 0$ for $x > 1$. So $f$ increases on $x < 1$, and decreases on $x > 1$. The is a relative max at $(1,e^{-1})$
7,8) Since \( f''(x) = (-1)e^{-x} - (1-x)e^{-x} = (x-2)e^{-x} \), \( x = 2 \) is an inflection point, because \( f''(x) > 0 \) for \( x > 2 \), and \( f''(x) < 0 \) for \( x < 2 \). So \( f \) is concave up on \( x > 2 \), and concave down on \( x < 2 \).

The function looks like:
Let \( f(x) = x^2 \ln x \). Find out everything about this function and then graph it.

1) Domain: \( x > 0 \)

2) Intercepts: Note: \( \lim_{x \to 0^+} x^2 \ln x = 0 \).

\( x \) intercepts come from solving \( x^2 \ln x = 0 \), so \( x = 1 \).

3) Symmetry: No Symmetry

4) Asymptotes: Since \( \lim_{x \to \infty} f(x) = \infty \), there are no asymptotes.

5,6) Intervals of Increase and Decrease. Rel. Extrema

Since \( f'(x) = 2x \ln x + x^2 \frac{1}{x} = x(2 \ln x + 1) \), \( x = e^{-1/2} \) is a stationary point. Also, \( f''(x) < 0 \) for \( x < e^{-1/2} \), and \( f''(x) > 0 \) for \( x > e^{-1/2} \). So \( f \) increases on \( x > e^{-1/2} \), and decreases on \( x < e^{-1/2} \). The is a relative min at \( \left( e^{-1/2}, -\frac{1}{2} e^{-1} \right) \).
7,8) Since \( f''(x) = (2 \ln x + 1) + \frac{2}{x} = 2 \ln x + 3 \), \( x = e^{-3/2} \) is an inflection point, because \( f''(x) > 0 \) for \( x > e^{-3/2} \), and \( f''(x) < 0 \) for \( x < e^{-3/2} \). So \( f \) is concave up on \( x > e^{-3/2} \), and concave down on \( x < e^{-3/2} \). The point of inflection is \( \left( e^{-3/2}, -\frac{3}{2} e^{-3} \right) \).

The function looks like: