Antiderivatives

Any function $F(x)$ with the property that $F'(x) = f(x)$ on an interval $I$ is called an antiderivative of $f(x)$ on the interval $I$.

Examples:

a) $F(x) = x^2$ is an antiderivative of $f(x) = 2x$, since
   \[ F'(x) = \frac{d}{dx}(x^2) = 2x = f(x) \]

b) $G(t) = e^{6t}$ is an antiderivative of $g(t) = 6e^{6t}$, since
   \[ G'(t) = \frac{d}{dt}(e^{6t}) = 6e^{6t} = g(t) \]

c) $F(x) = \sin(2x)$ is an antiderivative of $f(x) = 2\cos(2x)$, since
   \[ F'(x) = \frac{d}{dx}(\sin(2x)) = 2\cos(2x) = f(x) \]

Note: We usually associate the uppercase-letter function as antiderivative of corresponding lowercase-letter function.
Examples:  

a) \( F(x) = x^6 + x^5 + x^4 \) is an antiderivative of \( f(x) = 6x^5 + 5x^4 + 4x^3 \), since \[ F'(x) = \frac{d}{dx}(x^6 + x^5 + x^4) = (6x^5 + 5x^4 + 4x^3) = f(x) \]

b) \( F(x) = e^{\sin x} \) is an antiderivative of \( f(x) = \cos x e^{\sin x} \), since \[ F'(x) = \frac{d}{dx}(e^{\sin x}) = (\cos x e^{\sin x}) = f(x) \]

c) \( D(t) = \frac{3}{2} t^4 + \sqrt{t} + 1 \) is an antiderivative of \( d(t) = 6t^3 + \frac{1}{2\sqrt{t}} \), since \[ D'(t) = \frac{d}{dt}\left(\frac{3}{2} t^4 + \sqrt{t} + 1\right) = 6t^3 + \frac{1}{2\sqrt{t}} = d(t) \]
Theorem 1: If $F(x)$ is an antiderivative of $f(x)$ on an interval $I$, and $C$ is any constant, then $F(x) + C$ is the most general antiderivative of $f(x)$ on $I$.

Finding Formulas for Antiderivatives of Power functions

When taking the derivative of a power function $f(x) = x^r$ we i) multiply down the exponent then, ii) subtract 1 from the exponent. To undo this we first add 1 to the exponent then divide by the new exponent: In other words:

Let $F(x) = \frac{x^{k+1}}{k+1} + C$, where $k \neq -1$ is any real number, and $C$ is an arbitrary constant. Then, $F(x)$ is an antiderivative for the function $f(x) = x^k$. Every antiderivative of $f(x)$, where $k \neq -1$ has this form.

This is because $F'(x) = \frac{d}{dx}\left(\frac{x^{k+1}}{k+1}\right) = \frac{(k+1)x^k}{k+1} = x^k = f(x)$
Examples:  

a) Find the antiderivative for the function $f(x) = x^3 + x^2$.

$$F(x) = \frac{x^4}{4} + \frac{x^3}{3} + C$$

b) Find the antiderivative for the function $f(x) = \frac{1}{3}x^4 + \frac{1}{2}x^3$.

$$F(x) = \frac{x^5}{15} + \frac{x^4}{8} + C$$

c) Find the antiderivative for the function $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$.

Rewrite $f$ as $f(x) = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$, and $F(x) = \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

For the case of where $k = -1$, that is:

$$f(x) = \frac{1}{x} = x^{-1}$$

The antiderivative is

$$F(x) = \ln x + C.$$
Antiderivatives of Sinusoidal Functions

Recall: \((\sin x)' = \cos x\) and \((\cos x)' = -\sin x\)

Let \(C\) be and constant

If \(f(x) = \sin x\), then \(F(x) = -\cos x + C\) is an antiderivative of \(f\).

If \(g(x) = \cos x\), then \(G(x) = \sin x + C\) is an antiderivative of \(g\).
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<td>( c f(x) )</td>
<td>( cF(x) )</td>
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<td>( e^x )</td>
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<td>( \sin x )</td>
<td>( -\cos x )</td>
<td>( \frac{1}{x\sqrt{x^2-1}} )</td>
<td>( \sec^{-1} x )</td>
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Remember!!

If \( f(x) = \sin \omega x \), then \( F(x) = -\frac{1}{\omega} \cos \omega x + C \) is an antiderivative of \( f \).

If \( g(x) = \cos \omega x \), then \( G(x) = \frac{1}{\omega} \sin \omega x + C \) is an antiderivative of \( g \).

If \( h(x) = e^{kx} \), then \( H(x) = \frac{1}{k} e^{kx} + C \) is an antiderivative of \( h \).

Examples:

\[
\begin{align*}
  f(x) &= \sin 3x \quad \text{then} \quad F(x) = -\frac{1}{3} \cos 3x + C \\
  g(x) &= \cos \pi x \quad \text{then} \quad G(x) = \frac{1}{\pi} \sin \pi x + C \\
  h(x) &= e^{-4x} \quad \text{then} \quad H(x) = -\frac{1}{4} e^{-4x} + C
\end{align*}
\]
Rectilinear Motion

**Example:** Suppose a particle moves in a straight line with acceleration given by \( a(t) = 2t^2 \) m/s\(^2\). If the particle starts at position \( s(0) = 2 \) m with velocity \( v(0) = -3 \) m/s, what is its position at any given time?

**Solution:**
Since \( s''(t) = a(t) = 2t^2 \), we find the antiderivative of \( s''(t) \), which is the velocity \( s'(t) = \frac{2}{3}t^3 + C \). Since the initial velocity is \(-3\), \( s'(0) = C = -3 \).

Finding the antiderivative of \( s'(t) \) gives the position \( s(t) \). Since \( s'(t) = \frac{2}{3}t^3 - 3 \), we calculate the antiderivative to be \( s(t) = \frac{1}{6}t^4 - 3t + C \).

Using \( s(0) = 2 \), gives \( s(t) = C = 2 \), so \( s(t) = \frac{1}{6}t^4 - 3t + 2 \).
**Example:** Suppose a particle is moving on a linear track so that its acceleration (in m/s²) is given by:

\[ a(t) = 2t^3 + e^{4t} + 2\sin 2t, \text{ for } t \in [0, 10], \]

where \( t \) is time in seconds. If the particle’s initial position is \( s(0) = 4 \) m, and initial velocity is \( v(0) = 5 \) m/s, what is its position at any given time?

**Solution:**

\[ s''(t) = a(t) \quad \text{so} \quad s''(t) = 2t^3 + e^{4t} + 2\sin 2t \]

Finding the antiderivative gives

\[ s'(t) = \frac{1}{2}t^4 + \frac{1}{4}e^{4t} - \cos 2t + C \]

\[ v(0) = 5 \quad \Rightarrow \quad s'(0) = 5 \quad \text{so} \]

\[ \frac{1}{4} - 1 + C = 5 \quad \Rightarrow \quad C = \frac{23}{4} \]
And so

\[ s'(t) = \frac{1}{2} t^4 + \frac{1}{4} e^{4t} - \cos 2t + \frac{23}{4} \]

Integrating again gives

\[ s(t) = \frac{1}{10} t^5 + \frac{1}{16} e^{4t} - \frac{1}{2} \sin 2t + \frac{23}{4} t + D \]

\[ s(0) = 4 \text{ means } \frac{1}{16} + D = 4 \text{ so } D = \frac{63}{16} \text{ and} \]

\[ s(t) = \frac{1}{10} t^5 + \frac{1}{16} e^{4t} - \frac{1}{2} \sin 2t + \frac{23}{4} t + \frac{63}{16} \]
We can now derive the equation of motion for an object traveling vertically in the Earth’s gravitational field. Assume an initial position of $h(0) = h_0$, and an initial velocity of $h'(0) = v(0) = v_0$.

If the only acceleration the particle experiences is the gravity, and if $h$ is measure positive upwards,

$$h''(t) = -g$$

The antiderivative here is:

$$h'(t) = -gt + C$$

Since $h'(0) = v(0) = v_0$, $C = v_0$  

$$h'(t) = -gt + v_0$$

Anti-differentiating again gives

$$h(t) = -\frac{1}{2}gt^2 + v_0t + C$$

Since $h(0) = h_0$, $C = h_0$, and

$$h(t) = -\frac{1}{2}gt^2 + v_0t + h_0$$