Introduction

In this document we demonstrate how to value bonds. In addition, we show that as we hold bonds from year to year, bond prices fluctuate. We show how to calculate holding period rates of returns (annualized), which is the preferred way to compare performance across various bonds and to other financial assets. Read the documents “Compounding and Discounting” and “Return Calculations” to familiarize your knowledge of time value and return calculation concepts.

We will start our discussion by defining the coupon bond. Suppose that the bond is issued at time $t = 0$ and will mature $n$ years later. The firm issues the bond in a primary market using an investment bank, and similar to an IPO, the “market” will determine the price of that bond. The bond price will be based on:

1. **maturity** The number of years before the bond is redeemed
2. **face value** A fixed amount per bond which is returned to the investor at maturity. It is also known as the par value.
3. **coupon rate** This determines the coupon payments which are period payments made to the investor.

Other factors which will influence the bond price are the possibility of a default by the firm (bond ratings), tax treatment (municipal bonds), and other provisions of the issue (bond indenture).

For simplification, let us assume that we are considering a government issued bond (Treasury bond) which does not have a default risk, and ignore taxes and bond indenture complications. For further simplification, let us assume that coupon payments are made annually.

The coupon payment equals the coupon rate times the face value. For example, for a $1000 face value, if the coupon rate is 5%, the coupon rate is $1000(0.05) = $50 per period. Suppose that this bond is issued at $t = 0$, and will mature at $t = 10$, the cash flows would look as follows:

![Cash Flows Diagram](image-url)
A bond is essentially a way for firms to borrow money from investors. You can think of the face value as your principal, and the coupon payments as interest payments. This is only accurate if the price is also equal to the face value which does not have to be the case. When would it be different, and how can investors ascertain the correct value for the bond?

For the first level of analysis, let us assume that at any given point in time, a risk-free rate of return is appropriate for all future cash flows. Since we assumed that the bonds have no default risk, we will discount at the risk-free rate $r$. You did all the problems in your first homework assignment under this assumption. In fact, using the material in “Compounding and Discounting”, the present value of the cash flows for any bond with face value of $1000, coupon payment $CP$, and discount rate $r$, would be:

$$PV_0 = \sum_{t=1}^{n} C_t (1 + r)^{-t} = CP \left[ \frac{1 - (1 + r)^{-(n-1)}}{r} \right] + [1000 + CP](1 + r)^{-n}$$

The cash flows are: $n$-1 cash flows of CP, and one final cash flow at $t = n$ of the 1,000 plus the final coupon payment.

**Example 1**: What is the present value of the ten year 5% coupon bond if the discount rate is 4%?

$$PV = 50 \left[ \frac{1 - (1.04)^{-9}}{.04} \right] + 1050(1.04)^{-10}$$

$$= 50(7.4353) + 1050(0.6756)$$

$$= 371.77 + 709.34 = 1081.11$$

As you can see, the present value of this bond is greater than $1,000. One way to explain this is to focus on the coupon payments that are 5% of the face value. Since you are only looking for 4% from this bond, you are willing to pay more money today for the bond than what you will receive back at $t = 10$.

**Note**: See the spreadsheet “Bond Prices and Returns.xls” which shows the same examples as in this document using a spreadsheet. We broke the cash flows up above as an annuity for all the coupons except the final cash flow plus the cash flow of the face value together with the final coupon payment to mimic the way you would solve bond problems on a spreadsheet.

For example 1, if cash flows are in cells C4:L4, the price of the bond would be available as =NPV(4%,C4:L4). Note that this is not exactly a “net” present value, but rather the present value of the cash flows between cell C4 and cell L4. We put
the formula into cell B4. For the 5% coupon rate example, we obtain $50 for every year between year t = 1 and t = 9, and in year t = 10 we obtain the face value of $1000 and the final coupon payment of $50.

You should now try to value the same bond using both 5% and 6% and you should get the following results:

@4%  PV = 1081.11 The bond is priced at a **premium**!
@5%  PV = 1000.00 The bond is priced at **par**!
@6%  PV = 926.40 The bond is priced at a **discount**!

If the discount rate is less than the coupon rate, the bond will be priced at a premium. If the discount rate is greater than the coupon rate, the bond will be priced at a discount. If the discount rate is equal to the coupon rate, the bond is priced at par (equal to face value). Of course you can do the calculations using the spreadsheet or using the above formulas, and the answer should be identical. For example, if you use the 5% discount rate:

\[
PV = \frac{1 - (1.05)^{-9}}{.05} + 1050(1.05)^{-10}
\]

\[
= 50(7.10782) + 1050(0.61391)
\]

\[
= 355.39 + 644.61 = 1000
\]

Now consider holding this bond for one year. You will now be at t = 1, and the maturity of the bond is now nine years. If the discount rate should still be 5%, it should be obvious that the present value at t = 1 will still be $1,000. Example 2 below demonstrates this result:

**Example 2:** What is the present value of a nine year bond with 5% coupon payments when the discount rate is 5%?

\[
PV = \frac{1 - (1.05)^{-8}}{.05} + 1050(1.05)^{-9}
\]

\[
= 50(6.46321) + 1050(0.64461)
\]

\[
= 323.16 + 676.84 = 1000
\]

Suppose you purchases the above bond at t = 0 and now want to sell it at t = 1 for $1000, and want to calculate your holding period return for the year? It is made up of two components. One is the capital gain or loss on the bond price, and the other is the $50
coupon payment. Clearly the capital gain is zero since the price has not changed. The coupon payment of $50 provides for a 5% rate of return on the $1000 bond price. So without a change in interest rates, the promised rate of 5% as implied by the coupon rate is realized as a 5% holding period return.

On the spreadsheet, we call the Jan-96 time period \( t = 0 \) in example 2, because the person buying the bond on that date would think of it as the beginning of his bond. To the seller, it would at \( t = 1 \), because he is one year into holding this bond.

Now we will consider what a change in interest rates does to bond prices. We will assume that the interest rate jumps abruptly on January 1996 to 10%. Newly issued bonds with 9 year maturity are being sold with 10% coupon and are priced at par. Your “used” bond, which only provides a 5% coupon, could only be sold if they also yield 10% to investors. What should your bond be worth? Example 3 demonstrates below how investors would value your “used” bond using the new market interest rate of 10%:

**Example 3:** You could again find the present value but use nine years to maturity and 10% discount rate:

\[
P = 50 \left[ \frac{1 - (1.10)^{-8}}{.10} \right] + 1050 (1.10)^{-9}
\]

\[
= 50(5.334926198 ) + 1050 (0.424097618 )
\]

\[
= 266.7463099 + 445.3024993 = 712.05
\]

Notice the huge drop in the value of the bond. The marketplace is now demanding 10% for similar bonds. Your bond at the above price will provide the 10% return but only if it is priced at $712.05. We have demonstrated interest rate risk with this example. Of course, if interest rates go down, the bond would have increased in value. **Notice that the price of the bond is moving in the opposite direction to the change in interest rates.** When interest rate goes up, bond prices drop. When interest rates go down, bond prices increase.

How much was the loss to you, the old bondholder, over the year if interest rates moved from 5% to 10%? One good way to measure this is to calculate your overall rate of return. It would be the change in price between \( t = 0 \) and \( t = 1 \) plus the coupon payment over the initial price of the bond. One could say it is the sum of capital gains plus income, all calculated as a rate:

\[
r = \frac{(P_t + \text{Coupon Payment}) - P_0}{P_0} = \frac{712.05 + 50 - 1000}{1000} = -.23795
\]

Instead of making a 5% return, you would suffer a loss of almost 24%! We will now show that if the bond is purchased January 1996 for $712.05, and the interest remains at the new level of 10% over the next period through January 1997, the bond will indeed return 10% to the new bondholder, even though the coupon is 5%. You can think of the
buyer as the person you have sold your bond to for $712.05 when you took your loss in example 3 above. Remember that he could have purchased a newly issued bond at par paying a 10% coupon!

First, we need to price the bond January 1997, after the new buyer has held it for one year. At that point, the bond will have eight years until maturity:

**Example 5:** 8 year bond, coupon = $50, discount rate = 10%:

\[
P = 50 \left[ \frac{1 - (1.10)^{-8}}{.10} \right] + 1050 (1.10)^{-8}
\]

\[
= 50(4.868418818 ) + 1050 (0.46650738 )
\]

\[
= 243.4209409 + 489.8327492 = 733.25
\]

As you can see, it has come up slightly in price. Now we measure his return:

\[
r = \frac{(P_i + CouponPayment) - P_0}{P_0} = \frac{733.25 + 50 - 712.05}{712.05} = .09999
\]

This is essentially 10% (difference due to rounding). The bondholder receives the anticipated 10%, part of it through the 5% coupon, the rest through the capital appreciation of the bond. This is why the bond price had to drop when interest rates went up! It made the older bond competitive with other instruments assets that yielded 10%.

The point of these examples: As interest rates change, bond prices change to reflect the prevailing interest rates. Holders of existing bonds suffer the consequences (or the benefits since interest rates can also go down) but purchasers of bonds which are re-priced after the change in interest rate should obtain the implied holding period rate unless interest rates change again.

What if you never sell your bonds? You should end up with the rate that is implied when you purchase the bond, but in the above example that means 5% over the remaining years even though interest rates are now 10%. Clearly that is not ideal. The falling bond price reflects the new value of your bond.

As we demonstrated here, bonds are risky, and one risk they face is *interest rate risk*. But are some bonds more sensitive to changes in interest rates than other bonds? Let us stick with the 5% coupon - $1,000 face value bond. We showed the difference in price when the prevailing rate changed significantly on January 1996. In fact, we were showing the sensitivity of bond prices to interest rates for a bond which had nine years left before maturity. Instead of a nine year bond, let us repeat the same change in interest rates for a five year bond:

**Example 5:** 5 year bond, 5% coupon, discount rate 5%: PV of the bond = $1000
**Example 6:** Demonstrate that if interest rates move up to 10% when there is 5 years left to maturity, the bond price will be $810.46.

(Example 5 and 6 are not worked out here with the formulas. You should do it yourself, then check the spreadsheet where they do appear!)

Which of these two bonds (the nine year bond or the five year bond) is more sensitive to the change in interest rates? We need not consider a whole year’s holding period as before. We could just look at the price drop as the interest rate changes instantaneously. Normally, if the initial price is not identical, you would need to consider the drop in return form, but since both the nine and the five year bonds are priced at identical $1,000, we can just consider the price drop in dollars. The five year bond drops less in price when the interest rate changed! **It seems that the longer maturity bond is more sensitive to interest rate changes.**

There are two main characteristics of bonds which are looked at when considering sensitivity to interest rates. The first is maturity. As we have demonstrated, longer maturity bonds are more sensitive to changes in interest rates. Can you think intuitively why that should be the case? The second characteristic is coupon rates. **The homework on bonds asks you to compare a coupon bond with a zero-coupon bond.** You can think of the zero-coupon bond as certainly having a lower coupon rate!

A **zero-coupon bond** is a bond which does not pay coupons. To value the bond, you only need to discount the face value, so this is an easier calculation. With no coupons how to investors get compensated for lending money? It is all in the change in price from an obvious discounted price to the full face value. For example, a 10 year zero-coupon discount bond (or **pure discount bond**) has the following present value using a 5% rate:

\[
1000(1.05)^{-10} = 1000(0.613913254) = 613.91
\]

As you can see, zero-coupon bonds are easy to value since you only need to discount the face value. We won’t even bother with the spreadsheet for these.

If interest rates do not change, what should be the present value one year later?

\[
1000(1.05)^{-9} = 1000(0.644608916) = 644.61
\]

What about the return over that year? (No coupon payments!)

\[
r = \frac{(P_t + \text{CouponPayment}) - P_0}{P_0} = \frac{644.61 - 613.91}{613.91} = .05
\]

You should not be surprised at the answer. You received your 5% through the change in the bond price.
The homework for bonds has several learning objectives:

1. You will practice pricing both coupon bonds and zero-coupon bonds.
2. You will see what changing interest rates can do to prices.
3. You will demonstrate that bonds with lower coupons are more sensitive to interest rate risk.
4. You will see that the return to bonds comes from both the coupon payment and from changes in the bond price. This is similar to dividends and capital gains for stocks.

In the homework, your low coupon bond is the zero-coupon bond. Since they are always priced at a discount, you can not tell which bond is more sensitive by only looking at a price drop. **You must calculate actual returns.** As in most calculations related to investments, the dollar gain or loss is not as good a measure as a rate of return calculation! See the web document on return calculations.