Return Calculations

Introduction

In “Compounding and Discounting” we saw that given a cash flow at time $t = 0$, and an assumed interest rate, we could find the future value of that initial cash flow. Alternatively, we could take future cash flows (single or multiple) and find their present value. The typical story demonstrating the future value concept was the depositing money in a bank savings account. An application of the present value concept is taking a series of potential cash flows from a project and finding their present value at a given cost of capital and comparing it to $t = 0$ outflows to obtain a net present value. We are taking a single or multiple set of cash flows and an interest rate, and obtaining an equivalent cash flow at a different point in time (taking time value of money into account).

In this section, we will learn how to summarize a set of given cash flows which occur at different points in time with a single rate of return. In many ways, this is the same topic as the time value of money, but we now have all the cash flows and are solving for a rate, while before we had the rate (the interest rate) and solved for a cash flow. This duality will be made more explicit below, and it is especially clear for the single period analysis. The example we will use is a share of stock purchased at $t = 0$ and sold at $t = 1$, at which point we also obtain a possible dividend besides experiencing a gain (or a loss!) on the sale of the share.

Although there are several differences between buying a share of stock and putting your money in the bank, we want to focus on the similarities related to time value of money calculations and the return calculations for the stock purchase. As mentioned in “Compounding and Discounting”, it is important to get a feel for financial techniques and understand the connections rather than rely on a bunch of formulas.

Single Period Analysis

We can start making connections between various topics by considering only a single period in time from say $t = 0$ to $t = 1$. Consider buying a share of stock at $t_0$ for $10$, which we sell at $t_1$ for $11$. We represent prices for stocks at time $t$ as $P_t$.

Although we do not have to stick to a strict annual time scale for time value calculations, we found it easier to do so since interest rates are often quoted on an annual basis. But clearly we may hold a share of stock for a period much shorter or longer than a single year. We will call the period that we held on to the stock as the holding period. For stocks, the change in price can be positive or negative, and is also called capital gain or loss. The holding period capital gain is:

$$P_1 - P_0 = 11 - 10 = \$1.00$$
But for stocks another possible cash flow is a dividend payment. Let us assume that any dividends paid over the holding period are paid at the time the stock is sold to the owner of the stock over the holding period. Suppose .50 is paid as dividends. The total dollar return for this investment:

$$P_1 - P_0 + D = 11 - 10 + .50 = $1.50$$

For both these cases (with and without dividends), the dollar return might not be the best way to measure the performance of the stock. Perhaps the stock is an expensive stock and cost $100, or perhaps it only cost $10 as above. A dollar return of $1.50 is much more significant on the $10 stock, because nothing stops us from having purchased 10 shares instead of a single share. The same dollar return on the $100 share is not nearly as significant, although the initial investment is the same if we buy 10 shares of the $10 stock. A better measure to compare investments is the rate of return. The rate of return is calculated as the dollar return divided by the initial investment. For this situation:

$$r = \frac{(P_1 + D) - P_0}{P_0} = \frac{(11 + 0.50) - 10}{10} = 0.15$$

For percent return, multiply the rate of return by 100, or $(0.15)(100) = 15\%$. It doesn’t much matter which way you report this as long as you clearly label the percent as a percent. The advantage of using the rate of return measure over dollar return is that it is invariant to the amounts being invested and allows a better comparison of investment possibilities.
The most general way to write the return formula is as:

\[ r = \frac{C_1 - C_0}{C_0} \]

It is simply the change in cash flow over the initial cash flow. This is really the formula we should understand. We then use this for any situation that comes up, and we should not need a new formula to calculate the return for every possible financial asset. Instead, we could redefine our cash flows to reflect what they are in terms of prices, dividends, or any other cash flow. In this example we would include the dividend \( D_1 \) as part of \( C_1 \). Of course, we could write down the specific formula we just developed, but we shouldn’t have to memorize it.

\[ r = \frac{C_1 - C_0}{C_0} = \frac{(P_1 + D_1) - P_0}{P_0} = \frac{(11 + 0.50) - 10}{10} = 0.15 \]

Although the rate of return calculation is the best way to compare investments of different magnitudes, it does summarize two pieces of information (the cash flows) with a single number (the rate of return). There is a loss of information, which in this case is the magnitude of the money you made in dollar terms. The most basic information for any finance problem is the associated cash flows.

In the previous problems we have taken two cash flows at two points in time, and solved for the rate of return that the change in the value of the two cash flows represents. Suppose that a single cash flow is given, along with a rate of cash flow that is promised to us. We could then solve for the other cash flow. If we take our rate of return calculation, and solve for \( C_1 \):

\[
\begin{align*}
    r &= \frac{C_1 - C_0}{C_0} \\
    r &= \frac{C_1}{C_0} - \frac{C_0}{C_0} = \frac{C_1}{C_0} - 1 \\
    1 + r &= \frac{C_1}{C_0} \\
    C_1 &= C_0(1 + r)
\end{align*}
\]
You should recognize this as just the future value formula for a single period. If a promised return is 15%, a cash flow of $10 should turn into $11.50. You can also take the same rate of return formula and solve for \( C_0 \) instead of \( C_1 \) and obtain the present value formula:

\[
C_0 = \frac{C_1}{1 + r} = C_1 (1 + r)^{-1}
\]

The interpretation of the rate of return \( r \) could now be as the “required” rate we would demand on our investment, or the opportunity cost of investing our cash elsewhere. If \( C_1 \) is known with certainty today, we could interpret \( C_0 \) as the value of obtaining \( C_1 \), at the required rate \( r \). Since \( C_1 \) is known for certain, \( r \) should be the risk-free rate of return.

All this just demonstrates that present value, future value and return calculations are all related. Many of the problems we will consider are simple rate of return calculations for a single period. This includes the material on margin buying, short sales, and finding the return (or the index level) for a stock index like the DJIA.

### Annualizing Holding Period Returns

Consider a holding period return if it occurs over 3 months. Sometimes we want to compare various different holding period returns that occur over different time periods. We could standardize the different returns by asking what would be the equivalent return if we re-invest over a full year. We could set it up just as we described:

Suppose we call the holding period rate as \( r_h \). Then we are looking for \( r_a \) such that:

\[
1 + r_a = (1 + r_h)(1 + r_h) \cdots (1 + r_h)
\]

The number of terms on the right hand side would equal the number of times the holding period occurs in the year. Notice the similarity between this and the way we calculated the future value of an investment at \( t = 0 \): \( FV = C_0(1 + r)(1 + r)(1 + r) \cdots (1 + r) \). The sequence of \((1 + r)\) terms captured the compounding effect. We can simplify the above equation:

\[
1 + r_a = (1 + r_h)^{(\text{number of periods in a year})}
\]

---

1 These problems are considered in the next document “Returns on Indexes, Margin Accounts, and Short Sales”.

©2002 Steven Freund
Similar to the \( r_{EAR} \) problem, the future value should be identical using either the holding period repeatedly or the annualized return just once. In fact, that is how we set up the equality in the first place.

Example: Suppose you earned the 15% on your share of stock during a six month period. What is your annualized rate of return?

Solution:

\[
1 + r_a = (1 + r_h)^{\text{(number of periods in a year)}} = (1.15)^2
\]

\[
r_a = (1.15)^2 - 1 = 0.3225 \text{ or } 32.25\%
\]

Notice that this is greater than two times 15%. This is because of the compounding effect. But we are also assuming that we can make the same 15% return for both six month periods, which might not be true. The holding period does not need to be evenly divided in the year. In this case your exponent will not be an integer, but that is not a problem. Your calculator can still calculate \( y^x \), even if \( x \) is not a whole number.

**Geometric Mean**

The geometric mean is the average that we could use to replace a sequence of individual averages and still obtain the same future value. The geometric mean is applicable for any set of returns and is not tied to an annual measure. We simply ask what single return we could use to substitute a set of returns and still maintain the compounding effect present with the individual returns. Once we know what we want, it is easy to set up:

\[
[1 + r_g]^n = [(1 + r_1)(1 + r_2)....(1 + r_n)]
\]

Algebra yields the solution for \( r_g \), but more understanding is available from the above setup than the final form. To solve for \( r_g \), you need to get eliminate the exponent \( n \). If you recall,

\[
(a^b)^c = a^{bc}
\]

You want to eliminate the exponent \( n \) on the left hand side, so if you raise both sides to \( 1/n \), the left hand exponent becomes 1. It is actually better to understand how to do this algebra, because it is always easy to set up as above and it provides more understanding then the solution after the algebra.
\[ r_g = \left(1 + r_1 \right) \left(1 + r_2 \right) \ldots \left(1 + r_n \right)^{1/n} - 1 \]

Example: Suppose a 25% decrease is followed by a 25% increase. The arithmetic average of these two returns is 0%. But the future value of $100 which experiences these returns will not be $100.

\[ 100(1 - .25)(1 + .25) = 93.75 \]

Solution: The geometric mean is the rate which would produce the same 93.75 future value:

\[ 100(1 + r_g)^2 = 93.75 \]

Notice that we don’t really need the future value 93.75 since we can set each side equal to each other, nor do we need initial cash flow $100, since it can be eliminated from each side. I started with these to help you understand what is going on, but we can just equate the two ways the returns combine, which is what I had at the beginning of the section.

\[ (1 + r_g)^2 = (.75)(1.25) \]

\[ r_g = [(0.75)(1.25)]^{1/2} - 1 = 0.968245837 - 1 = -0.031754163 \]

check: \[ 100[1 + (-0.031754163)][1 + (-0.031754163)] = 93.75 \]

This problem seems a bit more complicated, because one of the returns is negative. A simpler geometric problem is illustrated in the appendix, where the topic is on calculator computations and accuracy. I presented this problem because I wanted to focus on negative returns. You might have noticed that the geometric problem also has a sequence of terms which are equal to one plus a return. Since the sequence comes up quite frequently, it is worth discussing a bit more. In the simple return calculation, we can express the return as the difference between a future price and the present price, divided by the present price. We can also use prices to indicate the quantity one plus the return:

\[ r = \frac{P_1 - P_0}{P_0} = \frac{P_1}{P_0} - 1 \]

Therefore

\[ 1 + r = \frac{P_1}{P_0} \]
The quantity $1 + r$ can be written as the ratio of future and present prices (or cash flows) and it is sometimes called the price relative. It is sometimes advantages to just calculate price relatives instead of returns, particularly when we will continue to use the quantity $1 + r$. Under this situation, a negative return will yield a price relative less than 1, while a positive return yields a price relative greater than one.

It is important to have a good understanding of what happens when we re-invest cash flows. The understanding of the sequence of price relatives which are always combined through multiplication:

$$(1 + r_1)(1 + r_2)\ldots(1 + r_n)$$

can’t be overemphasized. This also leads to the compounding and discounting factors.

**Real Rates, Nominal Rates, and Inflation**

Our final example also fits this category. Although occurring over the same period, the real rate of return interacts with inflation in a manner that resembles the compounding effect.

$$(1 + r_r)(1 + i) = (1 + r_n)$$

In general, the nominal rate is the promised rate. With a positive rate of inflation, the real rate $r_r$ is going to be less. If you do the implied math above, it is easy to see that in addition to the real rate plus inflation, the nominal rate contains an additional term that is the product of the real rate times inflation:

$$(1 + r_r)(1 + i) = (1 + r_n)$$

$$1 + r_r + i + r_r i = 1 + r_n$$

$$r_r + i + r_r i = r$$

Some people ignore the cross product term $r_r i$, but it can be large when inflation is large. This cross-product term is a bit like the interest on interest in compounding, and will make the real rate even less!

**Example:** Suppose your nominal rate of return over a period is 60%, but inflation over the same period is 50%.

**Solution:** If you ignore the cross product term, you would say that the real rate of interest is 10%. Using the above formula:
The real rate of return is only 6.67%. Either way, it is much less than 60% due to the high level of inflation.

**Accuracy and using a Calculator**

Most financial and accounting calculations are good enough if answers are rounded to the nearest penny, so often you set your calculator to show only two digits to the right of the decimal point. This level of accuracy is sometimes insufficient for returns. For example, consider a geometric average problem. Suppose a 10% increase is followed by a 20% increase. The geometric mean is calculated as:

\[
r_g = \left[ (1 + r_1)(1 + r_2) \right]^{1/2} - 1
\]

If our calculator rounds this to two places, it would give us 0.15 or 15%. **But this is identical to the arithmetic average.** If you allow this type of rounding, you would have rounded out the very difference between the arithmetic average and the geometric average. If you check if two consecutive returns of 15% will give you the same future value as 10% followed by 20%, it will not. This is particularly true if you have a large cash flow on which the return acts.

Although your calculator only shows the number of digits you want to see, it uses the full accuracy of its capability. Unfortunately this is destroyed if you take intermediate answers from the display, and put it back in a rounded form. It is suggested that you store intermediate answers in memory. It is also easy to change the display to show more digits to the right of the decimal point. Consult your calculator manual to make this change.

Rounding can be particularly distorting when you use the results in an exponent. Sometimes we need to take the cube root of a number, such as \( x^{1/3} \). But 1/3 is not exactly .33. It is more like .3333333333. This can make a big difference in your answers. One way to get around this is to input 3 into your calculator, and then use the 1/x key to obtain a more realistic value for 1/3. Or you could fill up the whole display with .3333333333.