General Guidelines

1. You may work in groups of up to four. Turn in just one paper for the whole group. Each member of the group will receive the same grade. If you would like to work in a group but cannot find a group to join, please let me know no later than Wednesday, September 17.

2. If you choose to turn in a revised version of your report, 90% of your grade will be based on the final version and 10% on the first version.

3. This is a major assignment, and you should plan to do some work on it every day, starting today. DO NOT WAIT UNTIL THE LAST MINUTE. IF YOU DO, YOU WILL NOT BE ABLE TO COMPLETE THE PROJECT.

4. The paper you turn in must be your group’s work. I reserve the right to ask any or all of you for a verbal explanation of your solution.

5. I strongly encourage you to see me regularly as you work on the project to discuss your progress. If you receive any assistance from anyone other than me, or if you use our textbook or any other outside source, you must cite the source in your paper. Remember guideline number 3.

What Should You Hand In?

After you have completely solved all parts of the assigned problem, you should write a report explaining your solution. The report should contain a mixture of words and equations and possibly graphs, tables, and diagrams. Your report should be grammatically correct, and you should use proper punctuation and spelling. If you do not type the paper, please write legibly. Your paper should contain an introduction explaining the problem and should clearly explain each step of your solution. Assume the reader knows something about vector calculus but has not read the project description below. You may find the following checklist helpful. It was adapted from a checklist developed by Dr. Annalisa Crannell of Franklin and Marshall College.

Does this paper

1. clearly restate the problem to be solved?
2. clearly label diagrams, tables, and graphs?
3. define all variables used?
4. provide a paragraph explaining how the problem will be approached?
5. explain how each formula is derived or give a reference indicating where it can be found?
6. give acknowledgment where it is due?

In this paper,

7. are the spelling, grammar, and punctuation correct?
8. is the mathematics correct?
9. did you answer all the questions that were asked?
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Presentation. (introduction, clarity and completeness of presentation, grammar) (10 points)  

Total.
Project Description

**Background Information** (See also Stewart, *Calculus: Concepts and Contexts*, pp. 731 - 733 and 735 - 736.)

The purpose of this project is to model the motion of a planet around the sun and to verify that Kepler’s Laws hold:

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.

2. The line joining the sun to a planet sweeps out equal areas in equal times.

3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

We use Newton’s Law of Gravitation as our starting point. To simplify the analysis, we assume that the sun and planet are point masses, we assume the sun is stationary, and we ignore the gravitational effects of all bodies except the sun on the planet’s motion.

We choose the origin of our coordinate system to be the position of the sun. We let \( t \) denote time, and we let \( \mathbf{r}, \mathbf{v}, \) and \( \mathbf{a} \) denote the planet’s position, velocity, and acceleration vectors, respectively. (\( \mathbf{r}, \mathbf{v}, \) and \( \mathbf{a} \) are functions of \( t \). \( \mathbf{v} = \mathbf{r}' \) and \( \mathbf{a} = \mathbf{v}' = \mathbf{r}'' \), where the prime denotes a derivative with respect to \( t \).) The scalar quantities \( r \) and \( v \) denote the magnitudes of \( \mathbf{r} \) and \( \mathbf{v} \), respectively. According to Newton’s Law of Gravitation, the gravitational acceleration experienced by the planet is

\[
\mathbf{a} = -\frac{GM}{r^3} \mathbf{r}
\]

where \( G \) is the gravitational constant and \( M \) is the mass of the sun.

Let \( \mathcal{P} \) denote the plane that contains the origin and the vectors \( \mathbf{r}(0) \) and \( \mathbf{v}(0) \). (We assume that the initial velocity vector \( \mathbf{v}(0) \) is pointing neither directly toward nor directly away from the sun.) Since there is no component of velocity or acceleration perpendicular to \( \mathcal{P} \) initially (recall that \( \mathbf{a} \) is in the \( -\mathbf{r} \) direction), the planet will remain confined to plane \( \mathcal{P} \) for all time. (You will show this more formally later.) We choose our coordinate axes so that \( \mathcal{P} \) is the xy plane. Using polar coordinates, we can express the planet’s position vector as \( \mathbf{r} = r \cos(\theta) \mathbf{i} + r \sin(\theta) \mathbf{j} \). We therefore have two unknown functions of \( t \): \( r \) and \( \theta \). To solve for these two functions we need two equations. We will use the equations of conservation of angular momentum and conservation of energy.
1. In parts a - i you will derive some useful facts for future use, and in part j you will establish Kepler’s first law.

(a) Show that \( \frac{d}{dt} (r \times v) = r \times a \).

(b) Let \( h = r \times v \). Show that \( h \) is constant by showing that its time derivative is 0. (Since \( h \) is perpendicular to both \( r \) and \( v \), the facts that \( h \) is constant and that \( h(0) \) is perpendicular to \( P \) mean that \( r \) and \( v \) are always contained in \( P \). Therefore, the planet is confined to the plane \( P \). Since \( mh \) is the planet’s angular momentum, where \( m \) denotes the mass of the planet, the condition that \( h \) is constant is equivalent to conservation of angular momentum.)

(c) Use the definitions \( h = r \times v \) and \( r = r \cos(\theta) \, i + r \sin(\theta) \, j \) to show that \( h = r^2 \theta' \, k \). (The prime denotes a derivative with respect to \( t \).)

(d) Take the time derivative of the equation \( r \cdot r = r^2 \) to show that \( r \cdot v = rr' \).

(e) Let \( E = \frac{1}{2} v^2 - \frac{GM}{r} \). Show that \( E \) is constant by showing that its time derivative is 0. Hint: \( v^2 = v \cdot v \). (\( mE \) is the planet’s energy, so the condition that \( E \) is constant is equivalent to conservation of energy.)

(f) Use the definition \( r = r \cos(\theta) \, i + r \sin(\theta) \, j \) to show that \( v^2 = (r')^2 + r^2 (\theta')^2 \). This means that \( E = \frac{1}{2} \left[ (r')^2 + r^2 (\theta')^2 \right] - \frac{GM}{r} \).

(g) We now have a system of two differential equations for \( r \) and \( \theta \):

\[
\begin{align*}
\frac{1}{2} (r')^2 + r^2 (\theta')^2 - \frac{GM}{r} & = E \\
\frac{r^2 \theta'}{h} & = h
\end{align*}
\]

We will look for a solution in which \( r \) is a function of \( \theta \). Use the chain rule to show that if \( r \) is a function of \( \theta \), the system becomes

\[
\begin{align*}
\frac{1}{2} \left[ (r')^2 + r^2 (\theta')^2 \right] - \frac{GM}{r} & = E \\
\left( \frac{dr}{d\theta} \right)^2 + r^2 & = cr^4 + dr^3 - r^2,
\end{align*}
\]

where \( c = 2E/h^2 \) and \( d = 2GM/h^2 \).
(i) Let \( u = 1/r \). Show that the differential equation from the previous step becomes

\[
\left( \frac{du}{d\theta} \right)^2 = c + du - u^2.
\]

Take the derivative of this equation to show that \( u \) must satisfy the second order equation

\[
\frac{d^2u}{d\theta^2} + u = c./2.
\]

Use Maple to solve this equation. (You should get \( u = C_1 \cos(\theta) + C_2 \sin(\theta) + \frac{c}{2} \).)

(j) We can orient our coordinate system so that \( C_2 = 0 \), giving us \( u = C_1 \cos(\theta) + \frac{c}{2} \). Show that

\[
r = \frac{p}{1 + \cos(\theta)}
\]

where \( p = 2/d = h^2/GM \) and \( e = 2C_1/d \). If \( 0 < e < 1 \), this is the polar equation of an ellipse with one focus at the origin. Thus, you have derived Kepler’s first law. (See Stewart, p. A73. The quantity \( e \) is called the eccentricity of the orbit. For the earth, \( e \approx 0.0167 \).)

2. As shown in Appendix H.2 of Stewart, \( dA/d\theta = r^2/2 \), where \( A \) denotes the area bounded by the \( x \) axis, the position vector \( r \), and the planet’s orbit. Use this fact and the chain rule to show that \( dA/dt = h/2 \), a constant. This is Kepler’s second law.

3. Carry out the following steps to demonstrate that Kepler’s third law is satisfied:

(a) Use Maple to integrate the equation \( dA/d\theta = r^2/2 \) (with \( r = p(1 + \cos(\theta)) \)) over the interval \( 0 \leq \theta \leq 2\pi \) to show that the area of the ellipse swept out by the planet is \( A = \frac{\pi p^2}{(1-e^2)^{3/2}} \). Hints: You will have to use the commands “assume” and “additionally” to tell Maple that \( 0 < e < 1 \). You will also have to use the ‘CauchyPrincipalValue’ option in the “int” command. See the Maple help utility for more details.

(b) Integrate the equation \( dA/dt = h/2 \) over the interval \( 0 \leq t \leq T \) (where \( T \) denotes the period of revolution of the planet) to obtain \( A = hT/2 \).

(c) Equate these two expressions for \( A \), solve for \( T \), and show that

\[
T^2 = \frac{4\pi^2}{GM} \left( \frac{p}{1-e^2} \right)^3.
\]

(Recall that \( p = h^2/GM \).)

(d) Finally, show that \( p/(1-e^2) = a \), where \( a \) denotes half the length of the major axis of the planet’s orbit. (Hint: Because of our choice of coordinate system, the major axis lies along the \( x \) axis. Find the coordinates of the points at which the planet’s orbit intersects the \( x \) axis, corresponding to \( \theta = 0 \) and \( \theta = \pi \). What is the length of the major axis?) Thus, \( T^2 \) is proportional to \( a^3 \), which is Kepler’s third law.