Problem 1. (15 pts.)
Solve the following initial value problem: \( y' + y = x, \ y(0) = 1 \).

This is a linear d.e., and it is already in standard form. 5 pts.

Find the integrating factor: \( \rho(x) = e^{\int 1 \, dx} = e^x \). 4 pts.

Multiply both sides of the standard form of the d.e. by the integrating factor:
\( e^x [y' + y] = xe^x \). 1 pt.

Use the Product Rule backwards to rewrite the d.e. as
\( \frac{d}{dx} [e^x y] = xe^x \). 2 pts.

Integrating both sides, we obtain
\( e^x y = \int xe^x \, dx = (x - 1)e^x + c \) using formula 46 from the Table of Integrals. 2 pts.

\( y(0) = 1 \Rightarrow e^0(1) = (0 - 1)e^0 + c \Rightarrow c = 2 \). 1 pt.

Therefore, \( e^x y = (x - 1)e^x + 2 \), so
\[ y = x - 1 + 2e^{-x}. \]

Problem 2. (10 pts.)
Solve the following initial value problem:
\( xy' + y^2 = xy^2, \ y(1) = 1 \)

\( xy' + y^2 = xy^2 \Rightarrow y' = \frac{2xy^2 - y^2}{x} = y^2 \left( \frac{2x - 1}{x} \right) \). This is a separable d.e. 3 pts.

Multiply by \( dx \) and divide by \( y^2 \):
\( \frac{dy}{y^2} = \left( \frac{2x - 1}{x} \right) \, dx \Rightarrow \int y^{-2} \, dy = \int \left( 2 - \frac{1}{x} \right) \, dx \Rightarrow y^{-1} = 2x - \ln(x) + c \). 4 pts.

\( y(1) = 1 \Rightarrow -1^{-1} = 2(1) - \ln(1) + c \Rightarrow c = -3 \). 1 pt.

Therefore, \( -y^{-1} = 2x - \ln(x) - 3 \Rightarrow y^{-1} = -2x + \ln(x) + 3 \Rightarrow y = [3 + \ln(x) - 2x]^{-1} \)

Problem 3. (10 pts.)
A tank initially contains 100 grams of a radioactive substance. After 1 hour there are 90 grams of the substance remaining in the tank. What is the half-life of the substance? In other words, when will there be 50 grams of the substance remaining in the tank?

Let \( t \) denote time (in hours) and let \( x \) denote the amount (in grams) of radioactive substance in the tank. Then \( x = x_0 e^{-kt} \) where \( x_0 = x(0) \). 5 pts.

\( x_0 = 100 \) so \( x = 100e^{-kt} \). 1 pt.

\( x(1) = 90 \Rightarrow 90 = 100e^{-k(1)} \Rightarrow 0.9 = e^{-k} \Rightarrow \ln(0.9) = \ln(e^{-k}) = -k \Rightarrow k = -\ln(0.9) \). 2 pts.

Let \( \tau \) denote the half-life. \( x(\tau) = 50 \Rightarrow 50 = 100e^{-k\tau} \Rightarrow 0.5 = e^{-k\tau} \Rightarrow \ln(0.5) = \ln(e^{-k\tau}) = -k\tau \Rightarrow \tau = -\frac{\ln(0.5)}{\ln(0.9)} \approx 6.6 \) hours. 2 pts.
Problem 4. (10 pts.) Find the general solution to each of the following linear homogeneous differential equations:

a. (5 pts.) \(y^{(4)} - 4y''' + 3y'' = 0\)

The characteristic equation is \(r^4 - 4r^3 + 3r^2 = 0 \Rightarrow r^2 (r^2 - 4r + 3) = 0 \Rightarrow r^2(r - 1)(r - 3) = 0 \Rightarrow r = 0, 1, 2\) \(2 \text{ pts.}\)

Therefore, \(y = c_1e^{0x} + c_2xe^{0x} + c_3e^{1x} + c_4e^{3x}\), or \(y = c_1 + c_2x + c_3e^x + c_4e^{3x}\) \(3 \text{ pts.}\)

b. (5 pts.) \(y'''' - 4y'' + 4y' = 0\)

The characteristic equation is \(r^3 - 4r^2 + 4r = 0 \Rightarrow r(r - 2)^2 = 0 \Rightarrow r = 0, 2\) \(2 \text{ pts.}\)

Therefore, \(y = c_1e^{0x} + c_2e^{2x} + c_3xe^{2x}\), or \(y = c_1 + c_2e^{2x} + c_3xe^{2x}\) \(3 \text{ pts.}\)

Problem 5. (15 pts.)

Solve the following initial value problem: \(y'' + y' - 2y = 8x^2\), \(y(0) = 4\), \(y'(0) = 0\).

Step 1. Find \(y_c\) by solving the d.e. \(y'' + y' - 2y = 0\).

Characteristic equation: \(r^2 + r - 2 = 0 \Rightarrow (r + 2)(r - 1) = 0 \Rightarrow r = -2, 1\).

Therefore, \(y_c = c_1e^{-2x} + c_2e^{x}\). \(3 \text{ pts.}\)

Step 2. Find \(y_p\). You can use either of the following methods.

Method 1: Undetermined Coefficients. The nonhomogeneous term in the d.e is \(8x^2\), a polynomial of degree 2. We should therefore guess that \(y_p\) is a polynomial of degree 2: \(y_p = Ax^2 + Bx + C\). No term in this guess duplicates a term in \(y_c\), so there is no need to modify this guess. \(3 \text{ pts.}\)

\(y = Ax^2 + Bx + C \Rightarrow y' = 2Ax + B \Rightarrow y'' = 2A\).

Therefore, the left side of the d.e. is \(y'' + y' - 2y = 2A + [2Ax + B] - 2 [Ax^2 + Bx + C] = -2Ax^2 + (2A - 4B)x + (2A + B - 2C)\).

We want this to equal the nonhomogeneous term \(8x^2\):

\[-2Ax^2 + (2A - 4B)x + (2A + B - 2C) = 8x^2 \Rightarrow -2A = 8, 2A - 4B = 0, 2A + B - 2C = 0\]

\(A = -4, B = -4, C = -6\). Thus, \(y_p = -4x^2 - 4x - 6\). \(6 \text{ pts.}\)

Method 2: Variation of Parameters. From \(y_c\) we obtain two independent solutions of the homogeneous d.e: \(y_1 = e^{-2x}\) and \(y_2 = e^{x}\). \(1 \text{ pt.}\)

The Wronskian is given by

\[W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & e^x \\ -2e^{-2x} & e^x \end{vmatrix} = e^{-2x}(e^x) - (-2e^{-2x}) = 3e^{-x}. \]

Therefore, \(y_p = u_1y_1 + u_2y_2\), \(y_p = \left( -\frac{1}{3} (4x^2 - 4x + 2) e^{2x} \right) e^{-x} + \left( -\frac{8}{3} (x^2 + 2x + 2) e^{-x} \right) e^x = -4x^2 - 4x - 6\) \(1 \text{ pt.}\)

Step 3. \(y = y_c + y_p\), so \(y = c_1e^{-2x} + c_2e^{x} - 4x^2 - 4x - 6\). \(1 \text{ pt.}\)

Step 4. Use the initial conditions to determine the values of \(c_1\) and \(c_2\). \(y = c_1e^{-2x} + c_2e^{x} - 4x^2 - 4x - 6 \Rightarrow y' = -2c_1e^{-2x} + c_2e^{x} - 8x - 4\)
Problem 6. (15 points)

Consider a forced, damped mass-spring system with mass 1 kg, damping coefficient 2 Ns/m, spring constant 9 N/m, and an external force \( F_{\text{ext}}(t) = 12 \cos(3t) \)N. Find the steady periodic solution (steady-state solution) for this system.

The d.e. modeling this system is \( m \ddot{x} + c \dot{x} + kx = F_{\text{ext}}(t) \), or \( \ddot{x} + 2\dot{x} + 9x = 12 \cos(3t) \). The steady-state (steady periodic) solution \( x_{sp} \) is the particular solution \( x_p \). You can find \( x_p \) using either the Method of Undetermined Coefficients the Method of Variation of Parameters. Here we use the Method of Undetermined Coefficients to save the work of finding \( x_c \).

Since the nonhomogeneous term in the d.e. \( 12 \cos(3t) \) is a cosine, we guess that \( x_p \) is the sum of a cosine and sine with the same frequency: \( x_p = A \cos(3t) + B \sin(3t) \). The complementary solution \( x_c \) will contain decaying exponential terms because of the damping term in the d.e., so we know that no term in our guess for \( x_p \) duplicates a term in \( x_c \). Therefore, there is no need to modify the guess. \( x = A \cos(3t) + B \sin(3t) \Rightarrow \dot{x} = -3A \sin(3t) + 3B \cos(3t) \Rightarrow x'' = -9A \cos(3t) - 9B \sin(3t) \).

Therefore, the left side of the d.e. is \( x'' + 2x' + 9x = -9A \cos(3t) - 9B \sin(3t) + 2 [-3A \sin(3t) + 3B \cos(3t)] + 9 [A \cos(3t) + B \sin(3t)] = 6B \cos(3t) - 6A \sin(3t) \). We want this to equal the nonhomogeneous term \( 12 \cos(3t) \), so \( 6B = 12 \) and \( -6A = 0 \Rightarrow A = 0 \) and \( B = 2 \). Therefore, \( x_{sp} = 2 \sin(3t) \).

Problem 7. (10 points)

a. (3 pts.) Find the Laplace transform of \( e^{-t} \cos(2t) \)

Using the Laplace transform table entry for \( \mathcal{L} \{ e^{at} \cos(bt) \} \) we have \( \mathcal{L} \{ e^{-t} \cos(2t) \} = \frac{s + 1}{(s + 1)^2 + 4} \)

b. (7 pts.) Find the inverse Laplace transform of \( \frac{s + 1}{s^2 - 3s + 2} \).

Use a partial fraction decomposition: \( \frac{s + 1}{s^2 - 3s + 2} = \frac{s + 1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2} \)

\((s-1)(s-2) \left[ \frac{s + 1}{(s-1)(s-2)} \right] = (s-1)(s-2) \left[ \frac{A}{s-1} + \frac{B}{s-2} \right] \Rightarrow s + 1 = A(s-2) + B(s-1) = (A + B)s + (-2A - B) \Rightarrow A + B = 1, -2A - B = 1 \Rightarrow A = -2, B = 3 \)

Therefore, \( \mathcal{L}^{-1} \left\{ \frac{s + 1}{s^2 - 3s + 2} \right\} = \mathcal{L}^{-1} \left\{ \frac{-2}{s-1} + \frac{3}{s-2} \right\} = -2 \mathcal{L}^{-1} \left\{ \frac{1}{s-1} \right\} + 3 \mathcal{L}^{-1} \left\{ \frac{1}{s-2} \right\} = -2e^t + 3e^{2t} \)

Problem 8. (15 points)

Use the Laplace Transform to solve the following IVP: \( x'' + x = 2e^t, \ x(0) = 1, \ x'(0) = 0 \). Solutions not using the Laplace transform method will not receive any credit.
\[ x'' + x = 2e^t \Rightarrow \mathcal{L}\{x'' + x\} = \mathcal{L}\{2e^t\} \Rightarrow \mathcal{L}\{x''\} - \mathcal{L}\{x'\} = 2\mathcal{L}\{e^t\} = \frac{2}{s-1} \quad \text{3 pts.} \]

\[ \Rightarrow \left[ s^2\mathcal{L}\{x\} - sx(0) - x'(0) \right] + \mathcal{L}\{x\} = \frac{2}{s-1} \quad \text{3 pts.} \]

\[ \left[ s^2\mathcal{L}\{x\} - s \cdot 1 - 0 \right] + \mathcal{L}\{x\} = \frac{2}{s-1} \Rightarrow \left(s^2 + 1\right) \mathcal{L}\{x\} = \frac{2}{s-1} + s = \frac{2 + s^2 - s}{s-1} = \frac{s^2 - s + 2}{s-1} \Rightarrow \]

\[ \mathcal{L}\{x\} = \frac{s^2 - s + 2}{(s-1)(s^2 + 1)} \quad \text{1 pt.} \Rightarrow x = \mathcal{L}^{-1}\left\{ \frac{s^2 - s + 2}{(s-1)(s^2 + 1)} \right\}. \]

Use a partial fraction decomposition:

\[ \frac{s^2 - s + 2}{(s-1)(s^2 + 1)} = \frac{A}{s-1} + \frac{Bs + C}{s^2 + 1} \]

\[ (s-1) \left(s^2 + 1\right) \left[ \frac{s^2 - s + 2}{(s-1)(s^2 + 1)} \right] = (s-1) \left(s^2 + 1\right) \left[ \frac{A}{s-1} + \frac{Bs + C}{s^2 + 1} \right] \Rightarrow \]

\[ s^2 - s + 2 = A \left(s^2 + 1\right) + (Bs + C) (s-1) = As^2 + A + Bs^2 + Cs - Bs - C = (A + B)s^2 + (C - B)s + (A - C) \]

\[ \Rightarrow A + B = 1, \quad C - B = -1, \quad A - C = 2 \Rightarrow A = 1, \quad B = 0, \quad C = -1. \quad \text{6 pts.} \]

Therefore, \[ x = \mathcal{L}^{-1}\left\{ \frac{1}{s-1} + \frac{0s - 1}{s^2 + 1} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{s-1} \right\} - \mathcal{L}^{-1}\left\{ \frac{1}{s^2 + 1} \right\} \Rightarrow x = e^t - \sin(t) \quad \text{2 pts.} \]