Problem 1. (10 pts.)

Solve the following initial value problem: \( y' = \frac{1}{3y^2(2x + 1)} \) with \( y(0) = 1 \). Note: \( y' \) means \( \frac{dy}{dx} \).

Problem 2. (10 pts.)

Solve the following initial value problem: \( y' = \frac{2x + y}{x} \), \( y(1) = 2 \).

Express your solution \( y \) explicitly in terms of \( x \). In other words, write your answer in the form \( y = \text{something} \). Note: \( y' \) means \( \frac{dy}{dx} \).

Problem 3. (15 points)

Let \( t \) denote time (in days) and let \( P \) denote the size of a mosquito population (in grams) at time \( t \). Suppose the daily birth rate per gram is \( \beta = 6 - 2P \), and suppose the daily death rate per gram is \( \delta = 2 \). (The units of \( \beta \) and \( \delta \) are (gram/day)/gram.)

a. (6 pts.) Write down the differential equation modeling this problem \( \frac{dP}{dt} = \text{something} \).

b. (6 pts.) Draw the phase line for the d.e. from part a.

c. (3 pts.) Suppose \( P(0) = 1 \). Use your phase line to find the limiting value of \( P(t) \) as \( t \) increases.

Problem 4. (10 points)

Find the general solution to each of the following differential equations.

a. (4 points) \( y'' + 2y' + 5y = 0 \)

b. (6 points) \( y^{(4)} + 5y^{(3)} + 6y'' = 0 \)
Problem 5. (15 points)
Solve the following initial value problem:

\[ y'' - y' - 2y = 4x - 8e^{3x} \quad \text{with } y(0) = 2 \text{ and } y'(0) = -8. \]

Note: \( y' = dy/dx \) and \( y'' = d^2y/dx^2 \)

Problem 6. (15 points)
Consider a damped, forced mass/spring system. Let \( t \) denote time (in seconds) and let \( x(t) \) denote the position (in meters) of the mass at time \( t \), with \( x = 0 \) corresponding to the equilibrium position. Suppose the mass \( m = 2 \) kg, the damping constant \( c = 8 \) N·s/m, the spring constant \( k = 6 \) N/m, and the external force is \( F_e(t) = 240 \cos(3t) \).

a. (13 pts.) Find the steady-state (steady periodic) solution \( x_{sp} \).

b. (2 pts.) Express your answer to part a in the form \( x_{sp} = C \cos(\omega t - \alpha) \)

Problem 7. (10 points)

a. (3 pts.) Find \( \mathcal{L} \{ \sqrt{t} + \sin(2t) \} \)

b. (7 pts.) Find \( \mathcal{L}^{-1} \left\{ \frac{8}{s^2 - 6s + 25} \right\} \).

Problem 8. (15 points)
Use the Laplace Transform to solve the following initial value problem:

\[ x'' + 4x = 10e^t \quad \text{with } x(0) = 0 \text{ and } x'(0) = 2. \]

Solutions to this IVP not using the Laplace transform method will not receive any credit. Primes denote derivatives with respect to \( t \): \( x' = dx/dt \) and \( x'' = d^2x/dt^2 \).