Exam 2 Review - Discrete Structures II - Spring 2018

Exam 2 is on Monday, March 26 in class and will cover Sections 7.1 through 7.3 plus the principle of inclusion-exclusion. No notes, textbook, calculator, or other electronic devices allowed! We will go over any questions you have from this sheet in class on Friday, March 23.

Review Problems:

1. Review the assigned homework problems from section 7.1 through 7.3, section 6.5, and the additional notes for the principle of inclusion-exclusion.

2. Be able to use the methodology of Section 7.1 to compute probability (i.e. use the \( p(E) = \frac{|E|}{|S|} \) formula).

3. Be able to state the principle of inclusion-exclusion for any number of sets.

4. Let \( A_1, A_2, A_3, A_4 \) be four sets satisfying the following conditions:
   - \( |A_1 \cup A_2 \cup A_3 \cup A_4| = 100 \)
   - \( |A_j \cap A_k| = 3k \) for all \( k > j \)
   - \( A_1 \cap A_2 \cap A_3 \cap A_4 = \emptyset \)

   Suppose that the cardinalities of the sets \( A_1, A_2, A_3, A_4 \) are all equal. Use the principle of inclusion-exclusion to find \( |A_1| \).

5. A coin is flipped seven times. What’s the probability that
   (a) exactly three tails occurs?
   (b) at most three heads occurs?
   (c) at most five heads occurs?
   (d) three heads occurs given that the first flip was heads?

6. Same question as the previous question, but assume the coin is biased so that probability of heads is 1/3. (You can leave your answers as a sum/difference/product/quotient of numbers.)

7. In a certain lottery, a subset of 4 different numbers is picked from the set \( \{1, 2, \ldots , 10\} \). A person wins $100 if they pick all four numbers correctly, $25 if they pick three out of four numbers correctly.
   (a) What’s the probability of winning $100?
   (b) What’s the probability of winning $25?
   (c) What’s the probability that a person picks a subset that does not contain any of the winning numbers?

8. An urn contains 3 red marbles, 2 yellow marbles, and 5 green marbles. Suppose you are blindfolded, and you pick three marbles from the urn, one at a time, without replacement. Let \( E_1 \) be the event that the first marble selected is red, and let \( E_2 \) be the event that the second marble selected is green.
   (a) Compute the following probabilities: \( p(E_2|E_1), p(E_1), p(E_2), p(E_1 \cap E_2) \), and \( p(E_1 \cup E_2) \).
   (b) What’s the probability that all three marbles are green?
   (c) What’s the probability that all three marbles are different colors?

9. Suppose the same urn from the previous problem is used. Four marbles are picked at random from the urn one at a time, with replacement (i.e. each marble chosen is put back into the urn before the next marble is selected at random). Let \( E_1 \) be the event that the first marble selected is red, and let \( E_2 \) be the event that the second marble selected is green.
   (a) Compute the following probabilities: \( p(E_2|E_1), p(E_1), p(E_2), p(E_1 \cap E_2) \), and \( p(E_1 \cup E_2) \).
   (b) What’s the probability all of the marbles are green?
   (c) What’s the probability that exactly two of the four choices are yellow?

10. Suppose that \( E, F \) are events in a sample space which satisfy \( p(E \cup F) = 0.5 \), \( p(E) = 0.3 \), and \( p(F) = 0.4 \). Compute \( p(E \cap F) \), \( p(E | F) \), and \( p(E \cap \overline{F}) \).
11. Suppose a die is loaded so that the probability of rolling an even number is twice as likely as rolling an odd number. Also, the probabilities of rolling the even numbers are all equal, and the probabilities of rolling the odd numbers are all equal. (The sides of the die are labeled with the numbers 1 through 6, as usual.)

(a) Find the probability distribution for the die.
(b) A pair of the loaded dice is thrown. What is the probability that
i. the numbers 2 and 3 are rolled on the first and second die, respectively?
ii. the sum of the dice is 10?
iii. one of the dice is even and the other is odd?
iv. the first die is less than 5?
v. the sum is 9 given that the first die is less than 5?

12. Suppose that 2% of people who do not use opium test positive for opium, and 5% of opium users test negative for opium. Furthermore, suppose that 1% of people actually use opium. Suppose that a person is selected at random. Let $N$ be the event that the person tests negative, and let $D$ be the event that the person uses opium. Compute the following probabilities. (You can leave your answers as a sum, different, product, and/or quotient of numbers.)

(a) $P(D)$
(b) $P(D \cap N)$
(c) $P(N \cap D)$
(d) $P(D|N)$
(e) $P(N|D)$
(f) $P(D|\overline{N})$
(g) $P(N)$
(h) the probability the person uses opium, given that it’s known they tested positive.
(i) the probability that the person uses opium and tested positive.
(j) the probability that the person tested negative or uses opium.

Answers:

4. 37
5. (a) 35/128
(b) 1/2
(c) 15/16
(d) 15/64
6. (a) $35 \cdot \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^4$
(b) $\left(\frac{2}{3}\right)^7 + 7 \cdot \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right) + 21 \cdot \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^2 + 35 \cdot \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^3$
(c) $1 - 7 \cdot \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right) - \left(\frac{1}{3}\right)^7$
(d) $15 \cdot \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4$
7. (a) 1/210
(b) 4/35
(c) 1/14
8. (a) $P(E_2|E_1) = 5/9, \ P(E_1) = 3/10, \ P(E_2) = 1/2, \ P(E_1 \cap E_2) = 1/6, \ P(E_1 \cup E_2) = 19/30$
(b) 1/12
(c) 1/4
9. (a) $P(E_2|E_1) = 1/2, \ P(E_1) = 3/10, \ P(E_2) = 1/2, \ P(E_1 \cap E_2) = 3/20, \ P(E_1 \cup E_2) = 13/20$
(b) 1/16
(c) 96/625
10. $P(E \cap F) = 0.2, \ P(E \mid F) = 0.2/0.4 = 0.5, \ P(E \cap \overline{F}) = 0.3 - 0.2 = 0.1$
11. (a) $P(1) = P(3) = P(5) = 1/9,$
(b) $P(2) = P(4) = P(6) = 2/9$
(i) 2/81
(ii) 1/9
(iii) 4/9
(iv) 2/3
(v) 2/27
12. (a) 0.99
(b) 0.0005
(c) 0.0095
(d) \( (0.05)(0.01) \)
(e) 0.05
(f) 0.95
(g) \( (0.05)(0.01) + (0.98)(0.99) \)
(h) \( (0.95)(0.01) + (0.02)(0.99) \)
(i) 0.0095
(j) $P(N \cup D) = (0.05)(0.01) + (0.98)(0.99) + 0.01 - 0.0005$
(coming inclusion-exclusion with parts (b) and (g))