1. Write the differential equations for the mechanical systems shown in Fig. 2.38.

**Figure 2.38**

Mechanical systems

![Mechanical systems diagram](image)

(a)

Solution:

The key is to draw the Free Body Diagram (FBD) in order to keep the signs right. For (a), to identify the direction of the spring forces on the object, let $x_2 = 0$ and fixed and increase $x_1$ from 0. Then the $k_1$ spring will be stretched producing its spring force to the left and the $k_2$ spring will be compressed producing its spring force to the left also. You can use the same technique on the damper forces and the other mass.

![Free Body Diagram](image)

(a)

$$m_1 \ddot{x}_1 = -k_1 x_1 - b_1 \dot{x}_1 - k_2 (x_1 - x_2)$$

$$m_2 \ddot{x}_2 = -k_2 (x_2 - x_1) - k_3 (x_2 - y) - b_2 \dot{x}_2$$
16. Write the dynamic equations and find the transfer functions for the circuits shown in Fig. 2.46.

(a) lead circuit
(b) lag circuit.
(c) notch circuit

Figure 2.46
Lead (a), lag (b), notch (c) circuits
Solution:

(a) lead circuit

\[
\frac{V_{in} - V}{R_2} + \frac{0 - V}{R_1} + C \frac{d}{dt} (0 - V) = 0 \tag{1}
\]

\[
\frac{V_{in} - V}{R_2} = \frac{0 - V_{out}}{R_f} \tag{2}
\]

We need to eliminate \( V \). From Eq. (2),

\[
V = V_{in} + \frac{R_2}{R_f} V_{out}
\]

Substitute \( V \)'s in Eq. (1),

\[
\frac{1}{R_2} \left( V_{in} - V_{in} - \frac{R_2}{R_f} V_{out} \right) - \frac{1}{R_1} \left( V_{in} + \frac{R_2}{R_f} V_{out} \right) - C \left( \dot{V}_{in} + \frac{R_2}{R_f} \dot{V}_{out} \right) = 0
\]

\[
\frac{1}{R_1} V_{in} + C \dot{V}_{in} = -\frac{1}{R_f} \left[ \left( 1 + \frac{R_2}{R_1} \right) V_{out} + R_2 C \dot{V}_{out} \right]
\]

Laplace Transform

\[
\frac{V_{out}}{V_{in}} = \frac{Cs + \frac{1}{R_1}}{-\frac{1}{R_f} \left( R_2 C s + 1 + \frac{R_2}{R_1} \right)} = -\frac{R_f}{R_2} \frac{s + \frac{1}{R_1 C}}{s + \frac{1}{R_1 C} + \frac{1}{R_2 C}}
\]

We can see that the pole is at the left side of the zero, which means a lead compensator.
(c) notch circuit

\[ C \frac{d}{dt} (V_{in} - V_1) + \frac{0 - V_1}{R/2} + C \frac{d}{dt} (V_{out} - V_1) = 0 \]

\[ \frac{V_{in} - V_2}{R} + 2C \frac{d}{dt} (0 - V_2) + \frac{V_{out} - V_2}{R} = 0 \]

\[ C \frac{d}{dt} (V_1 - V_{out}) + \frac{V_2 - V_{out}}{R} = 0 \]

We need to eliminate \( V_1, V_2 \) from three equations and find the relation between \( V_{in} \) and \( V_{out} \)

\[ V_1 = \frac{C_s}{2(C_s + \frac{1}{R})} (V_{in} + V_{out}) \]

\[ V_2 = \frac{\frac{1}{R}}{2(C_s + \frac{1}{R})} (V_{in} + V_{out}) \]

\[ C_s V_1 - C_s V_{out} + \frac{1}{R} V_2 - \frac{1}{R} V_{out} \]

\[ = C_s \frac{C_s}{2(C_s + \frac{1}{R})} (V_{in} + V_{out}) + \frac{1}{R} \frac{1}{2(C_s + \frac{1}{R})} (V_{in} + V_{out}) - \left( C_s + \frac{1}{R} \right) V_{out} \]

\[ = 0 \]
\[ \frac{C^2 s^2 + \frac{1}{R^2}}{2 \left( C s + \frac{1}{R} \right)} V_{in} = \left[ \left( C s + \frac{1}{R} \right) - \frac{C^2 s^2 + \frac{1}{R^2}}{2 \left( C s + \frac{1}{R} \right)} \right] V_{out} \]

\[ \frac{V_{out}}{V_{in}} = \frac{\frac{C^2 s^2 + \frac{1}{R^2}}{2 \left( C s + \frac{1}{R} \right)}}{\left( C s + \frac{1}{R} \right) - \frac{C^2 s^2 + \frac{1}{R^2}}{2 \left( C s + \frac{1}{R} \right)}} = \frac{\left( C^2 s^2 + \frac{1}{R^2} \right)}{2 \left( C s + \frac{1}{R} \right)^2 - \left( C^2 s^2 + \frac{1}{R^2} \right)} = \frac{C^2 \left( s^2 + \frac{1}{R^2} \right)}{C^2 s^2 + 4 \frac{C s}{R} + \frac{1}{R^2}} \]

\[ = \frac{s^2 + \frac{1}{R^2 C^2}}{s^2 + \frac{4}{R C} s + \frac{1}{R^2 C^2}} \]

26. For the two-tank fluid-flow system shown in Fig. 2.53, find the differential equations relating the flow into the first tank to the flow out of the second tank.

**Figure 2.53**
Two-tank fluid-flow system for Problem 2.26
Solution:

This is a variation on the problem solved in Example 2.20 and the definitions of terms is taken from that. From the relation between the height of the water and mass flow rate, the continuity equations are

\[ \dot{m}_1 = \rho A_1 \dot{h}_1 - w_{in} - w \]

\[ \dot{m}_2 = \rho A_2 \dot{h}_2 = w - w_{out} \]

Also from the relation between the pressure and outgoing mass flow rate,

\[ w = \frac{1}{R_1} (\rho gh_1)^{\frac{1}{2}} \]

\[ w_{out} = \frac{1}{R_2} (\rho gh_2)^{\frac{1}{2}} \]

Finally,

\[ \dot{h}_1 = -\frac{1}{\rho A_1 R_1} (\rho gh_1)^{\frac{1}{2}} + \frac{1}{\rho A_1} w_{in} \]

\[ \dot{h}_2 = \frac{1}{\rho A_2 R_1} (\rho gh_1)^{\frac{1}{2}} - \frac{1}{\rho A_2 R_2} (\rho gh_2)^{\frac{1}{2}}. \]