4.4 The DC-motor speed control in Fig. 4.38 is described by the differential equation
\[ \dot{y} + 60y = 600v_a - 1500w, \]
where \( y \) is the motor speed, \( v_a \) is the armature voltage, and \( w \) is the load torque. Assume the armature voltage is computed using the PI control law
\[ v_a = \left( k_p e + k_i \int_0^t e dt \right), \]
where \( e = r - y \).

(a) Compute the transfer function from \( W \) to \( Y \) as a function of \( k_p \) and \( k_i \).

(b) Compute values for \( k_p \) and \( k_i \) so that the characteristic equation of the closed-loop system will have roots at \(-60 \pm 60j\).

\[ \begin{align*}
W & \quad 1500 \\
\Sigma & \quad e^+ \\
D & \quad v_a \\
\Sigma & \quad 600 \\
\frac{1}{s + 60} & \quad Y \\
\end{align*} \]

Figure 4.38: Unity feedback system with prefilter for Problem 4.4

Solution:

(a) Transfer function: Set \( R(s) = 0 \), then \( E(s) = -Y(s) \)
\[ Y(s) = \frac{-600 \left( k_p + \frac{k_i}{s} \right) Y(s) - 1500W(s)}{(s + 60)} \]
\[ Y(s) = \frac{-1500s}{s^2 + 60(1 + 10k_p)s + 600k_i} \]

(b) For roots at \( s_{1,2} = -60 \pm j60 \),
\[ \begin{align*}
\{s_1 + s_2 &= -60(1 + 10k_p) = -120 \\
s_1s_2 &= 7200 = 600k_i \}
\end{align*} \]
\[ \Rightarrow \quad k_p = 0.1, \quad k_i = 12 \]

4.26 Consider the system shown in Fig. 4.49 with PI control.

(a) Determine the transfer function from \( R \) to \( Y \).

(b) Determine the transfer function from \( W \) to \( Y \).

(c) Use Routh’s criteria to find the range of \( (k_p, k_i) \) for which the system is stable.

(d*) Pick \( k_p \) and \( k_i \) so that the closed-loop system is stable. What is the steady state error when \( r(t) = 1(t) \) and \( w(t) = 1(t) \)?
Solution:

\[ \frac{Y(s)}{R(s)} = \frac{10 + \frac{k_p s + k_i}{s^2 + s + 20}}{1 + \frac{10 + \frac{k_p s + k_i}{s^2 + s + 20}}{s^3 + s^2 + 10(2 + k_p)s + 10k_i}} = \frac{10(k_p s + k_i)}{s^3 + s^2 + 10(2 + k_p)s + 10k_i} \]

\[ \frac{Y(s)}{W(s)} = \frac{\frac{s^2 + s + 20}{10}}{1 + \frac{10 + \frac{k_p s + k_i}{s^2 + s + 20}}{s^3 + s^2 + 10(2 + k_p)s + 10k_i}} = \frac{10s}{s^3 + s^2 + 10(2 + k_p)s + 10k_i} \]

(c) The characteristic equation is \( s^3 + s^2 + 10(2 + k_p)s + 10k_i = 0 \). The Routh's array is

\[
\begin{array}{ccc}
 s^3 & 1 & 10(2 + k_p) \\
 s^2 & 1 & 10k_i \\
 s^1 & 10(2 + k_p - k_i) \\
 s^0 & 10k_i \\
\end{array}
\]

For stability we must have \( k_i > 0 \) and \( k_p > k_i - 2 \).

\( \text{P}^* \) Choose \( k_i = 1, \ k_p = 1 \), The transfer functions become,

\[ \frac{Y(s)}{R(s)} = \frac{10s + 10}{s^3 + s^2 + 30s + 10}, \quad \frac{Y(s)}{W(s)} = \frac{10s}{s^3 + s^2 + 30s + 10} \]

Since \( \frac{Y(0)}{R(0)} = 1, \ \frac{Y(s)}{W(s)} = 0 \), the steady state error is 0.

As a verification, let \( r(t) = 1(t) \) and \( w(t) = 1(t) \), using matlab, we can get the following response,