A General Framework in State-Space Approach

Given an LTI system:
\[
\dot{x} = Ax + Bu; \quad y = Cx \quad (*)
\]
The system might be unstable or doesn’t meet the required performance spec. How can we improve the situation?

The main approach: Let \( u = v - Kx \) (state feedback), then
\[
\begin{align*}
\dot{x} &= Ax + B(v - Kx); \quad y = Cx + D(v - Kx) \\
&= (A - BK)x + Bv; \quad = (C - DK)x - Dv
\end{align*}
\]
The performance of the system is changed by matrix \( K \).

Questions:
- Is there a matrix \( K \) s.t. \( A - BK \) is stable?
- Can \( \text{eig}(A - BK) \) be moved to desired locations?

These issues are related to the controllability of \( (*) \)
Main Result 1: The eigenvalues of $A-BK$ can be moved to any desired locations iff the system (*) is controllable.

Another situation: the state $x$ is not completely available. Only a linear combination of $x$, e.g., $y = Cx$, can be measured. How can we realize $u = v-Kx$?

A possible solution: build an observer to estimate $x$ based on measurement of $y$.

Main result 2: The observer error (difference between the real $x$ and estimated $x$) can be made arbitrarily small within arbitrarily short time period iff (*) is observable.

We will arrive at these conclusions in Chapter 8. Before that, we need to prepare some tools and go through these fundamental problems: controllability and observability.

**Controllability: Definition**

Consider the system

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n; \quad u \in \mathbb{R}^p.$$  

Controllability is a relationship between state and input.

**Definition:** The system, or the pair $(A,B)$, is said to be controllable if for any initial state $x(0) = x_0$ and any final state $x_d$, there exist a finite time $T > 0$ and an input $u(t)$, $t \in [0,T]$ such that

$$x(T) = e^{AT}x_0 + \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau = x_d$$  \hspace{1cm} (1)

**Comment:** There may exist different $T$ and $u$ that satisfy (1). As a result, there may be different trajectories starting from $x_0$ and end at $x_d$. Controllability does not care about the difference.
\[ W_c(t) = \int_0^t e^{A\tau}BB'e^{A^\tau}d\tau = \int_0^t e^{A(t-\tau)}BB'e^{A(t-\tau)}d\tau \]

**Equivalent conditions:** The following are equivalent conditions for the pair \((A,B)\) to be controllable:

1) \( W_c(t) \) is nonsingular for every \( t > 0 \).
2) \( W_c(t) \) is nonsingular for at least one \( t > 0 \).
3) For every \( v \in \mathbb{R}^n, v \neq 0 \), \( v'e^{At}B \) is not identically zero.
4) The matrix \( G^c = [B\ AB\ A^2B\ \ldots\ A^{n-1}B] \) has full row rank, i.e., \( \rho(G^c) = n \).
5) The matrix \( M(\lambda) = [A-\lambda I\ B] \) has full row rank at all \( \lambda \in \mathbb{C} \).
6) \( M(\lambda) \) has full row rank at every eigenvalues of \( A \).

Note: \( M(\lambda) \) has full row rank if \( \lambda \) is not an eigenvalue of \( A \). We only need to check the rank of \( M(\lambda) \) at eigenvalues of \( A \).

Note: Of all the conditions, only 4) and 6) can be practically verified.

**Example:** Determine the controllability for

\[ \dot{x} = Ax + Bu, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} a \\ b \end{bmatrix} \]

**Approach 1:** \( G^c = [B\ AB] = \begin{bmatrix} a & -a \\ b & -b \end{bmatrix} \)

\( \rho(G) < 2 = n \) for all possible \( a \) and \( b \).

The system not controllable whatever \( a \) and \( b \) are.

**Approach 2:** Check \( M(\lambda) = [A-\lambda I\ B] \) at \( \lambda = -1 \)

\[ M(-1) = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \end{bmatrix} \]

\( \rho(M(-1)) < 2 \) for all possible \( a \) and \( b \).

Same conclusion on controllability.
Example:

\[ \dot{x} = Ax + Bu, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} a \\ b \end{bmatrix} \]

Approach 1: \( G^c = [B \ AB] = \begin{bmatrix} a & -a \\ b & -2b \end{bmatrix} \)

\[ \det G^c = -ab, \quad \begin{cases} \rho(G^c) = 2, & \text{if } a \neq 0 \text{ and } b \neq 0 \\ \rho(G^c) < 2, & \text{if either } a = 0 \text{ or } b = 0 \end{cases} \]

The system is controllable if \( a \neq 0 \) and \( b \neq 0 \).

Approach 2: Check \( M(\lambda) = [A-\lambda I \ B] \) at \( \lambda = -1 \)

\[ M(-1) = \begin{bmatrix} 0 & 0 & a \\ 0 & -1 & b \end{bmatrix}, \quad M(-2) = \begin{bmatrix} 1 & 0 & a \\ 0 & 0 & b \end{bmatrix}, \quad \rho(M(-1)) = \rho(M(-2)) = 2 \iff a \neq 0 \text{ and } b \neq 0 \]

Same conclusion on controllability

A general SI system (diagonalizable)

\[ \dot{x} = Ax + Bu, \quad A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \]

The above system is controllable if and only if the eigenvalues are distinct and none of the \( b_i \)'s is zero
Example:
\[
\dot{x} = Ax + Bu, \quad A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}
\]

\[
G^c = [B \ AB] = \begin{bmatrix} b_1 & \alpha b_1 - \beta b_2 \\ b_2 & \beta b_1 + \alpha b_2 \end{bmatrix}
\]

\[
\det G^c = \beta(b_1^2 + b_2^2), \quad \begin{cases} \rho(G^c) = 2, & \text{if } \beta \neq 0 \text{ and } b_1^2 + b_2^2 \neq 0 \\ \rho(G^c) < 2, & \text{if either } \beta = 0 \text{ or } b_1^2 + b_2^2 = 0 \end{cases}
\]

The system is controllable if \(\beta \neq 0\) and \((b_1, b_2) \neq (0,0)\)

Theorem: Consider the pair

\[
A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix}
\]

Suppose that the eigenvalues of \(A_i\) and those of \(A_j\) are disjoint for \(i \neq j\). Then \((A,B)\) is controllable iff \((A_i,B_1)\) is controllable for all \(i\).
**Theorem:** Let $\rho(B) = p$. The pair $(A,B)$ is controllable iff

$$G_{n-p+1}^c = [B \ AB \ A^2B \ldots A^{n-p}B]$$

has full row rank. This is equivalent to $G_{n-p+1}^c G_{n-p+1}^{c'}$ being nonsingular, and to $G_{n-p+1}^c G_{n-p+1}^{c'} > 0$ (positive definite.)

**Example:**

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$n=4$, $p=2$. $\rho(B) = 2 = p$.

$$G_{n-p+1}^c = [B \ AB \ A^2B] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 & 0 & -4 \\ b_1 & b_2 & Ab_1 & Ab_2 & A^2b_1 & A^2b_2 \end{bmatrix}$$

The first 4 columns are LI. $\Rightarrow \rho(G_{n-p+1}^c) = 4 = n$  
$\Rightarrow (A,B)$ controllable
Example:

\[
A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad n=4, \ p=2. \ \rho(B)=2.
\]

\[
G_{n-p+1}^c = \begin{bmatrix} B & AB & A^2B \\ b_1 & b_2 & Ab_1 & Ab_2 & A^2b_1 & A^2b_2 \end{bmatrix}
\]

The first 3 columns are LI.
The 4th is dependent on the first 3.

\[
\begin{bmatrix} b_1, b_2, Ab_1, A^2b_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 3 & 8 \\ 1 & 0 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{bmatrix} \quad \text{has full row rank}
\]

Hence \((A,B)\) is controllable.

**Effect of equivalence transformation**

Recall that equivalence transformation can make the structure cleaner and simplify analysis.

**Question:**
Does similarity transformation retain the controllability property?

**Theorem:** The controllability property is invariant under any equivalence transformation

**Proof:** Consider \((A,B)\) with \(G^c=[B \ AB \ A^2B \ \ldots \ \ A^{n-1}B]\).

Let the transformation matrix be \(P\). Then \((A,B) \Leftrightarrow (PAP^{-1}, PB)\)

\[
G^c = \begin{bmatrix} B & AB \ \ldots \ \ A^{n-1}B \end{bmatrix}
= \begin{bmatrix} PB & PAP^{-1}PB \ \ldots \ \ PA^{n-1}P^{-1}PB \end{bmatrix}
= \begin{bmatrix} PB & PAB \ \ldots \ \ PA^{n-1}B \end{bmatrix}
= PG^c
\]

Since \(P\) is nonsingular,

\[
\rho(G^c) = \rho(G^c)
\]
Observability: A dual concept

Consider an n-dimensional, p-input, q-output system:
\[
\dot{x} = Ax + Bu; \quad y = Cx + Du
\]

Assume that we know the input and can measure the output, but has no access to the state.

**Definition:** The system, is said to be observable if for any unknown initial state \( x(0) \), there exists a finite \( t_1 > 0 \) such that \( x(0) \) can be exactly evaluated over \([0,t_1]\) from the input \( u \) and the output \( y \). Otherwise the system is said to be unobservable.
Duality between controllability and observability

**Theorem of duality:** The pair \((A, B)\) is controllable if and only if \((A_1, C_1) = (A', B')\) is observable.

\[
\dot{x} = Ax + Bu \quad \text{Dual systems} \quad \dot{z} = A_1z = A'z
\]
\[
y = C_1z = B'z
\]

**Equivalent conditions for observability:**

1) The pair \((A, C)\) is observable.
2) \(W_0(t)\) is nonsingular for some \(t > 0\).
3) The observability matrix
\[
G^o = \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix}
\]
has full column rank, i.e., \(\rho(G^o) = n\).

4) The matrix
\[
M^o(\lambda) = \begin{bmatrix}
A - \lambda I \\
C
\end{bmatrix}
\]
has full column rank at every eigenvalue of \(A\).
Theorem: The pair \((A,C)\) is observable if and only if
\[
G_{n-q+1}^o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-q} \end{bmatrix}
\]
has full column rank, where \(q=\rho(C)\).

Theorem: The observability property is invariant under any equivalence transformation;

Theorem: Consider the pair
\[
A = \begin{bmatrix} A_i & 0 & \cdots & 0 \\ 0 & A_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 & \cdots & C_m \end{bmatrix}
\]

Suppose that the eigenvalues of \(A_i\) and those of \(A_j\) are disjoint for \(i \neq j\). Then \((A,C)\) is observable iff \((A_i,C_i)\) is observable for all \(i\).
So far, we have learned
• Controllability
• Observability
Next, we will study
• Canonical decomposition: to divide the state space into controllable/uncontrollable, observable/unobservable subspaces

**Canonical Decomposition**

Consider an LTI system,
\[ \dot{x} = Ax + Bu, \quad y = Cx + Du \]
Let \( z = Px \), where \( P \) is nonsingular, then
\[ \dot{z} = \bar{A}x + \bar{B}u, \quad y = \bar{C}z + \bar{D}u \]
where \( \bar{A} = PA^{-1}P, \quad \bar{B} = PB, \quad \bar{C} = CP^{-1}, \quad \bar{D} = D \)

Recall that under an equivalence transformation, all properties, such as stability, controllability and observability are preserved.

We also have \( \bar{G}^c = PG^c, \quad \bar{G}^o = G^oP^{-1} \)

Next we are going to use equivalence transformation to obtain certain specific structures which reflect controllability and observability.
Controllability decomposition

Recall \( G^c = [B \; AB \; \ldots \; A^{n-1}B] \). Suppose that \( \rho(G^c) = n_1 < n \). Then \( G^c \) has at most \( n_1 \) LI columns. They form a basis for the range space of \( G^c \):

**Theorem:** Suppose that \( \rho(G^c) = n_1 < n \). Let \( Q \) be a nonsingular matrix whose first \( n_1 \) columns are LI columns of \( G^c \). Let \( P = Q^{-1} \). Then

\[
\begin{align*}
\overline{A} &= PA^{-1} = \begin{bmatrix} \overline{A}_c & \overline{A}_{12} \\ \overline{0} & \overline{A}_2 \end{bmatrix}, \\
\overline{B} &= PB = \begin{bmatrix} \overline{B}_c \\ \overline{0} \end{bmatrix}, \\
\overline{C} &= \begin{bmatrix} \overline{C}_c \\ \overline{C}_2 \end{bmatrix}
\end{align*}
\]

Moreover, the pair \( (\overline{A}_c, \overline{B}_c) \) is controllable and

\[
\overline{C}_c(sI - \overline{A}_c)^{-1} \overline{B}_c + D = C(sI - A)^{-1}B + D
\]

See page 159 for the proof.

Discussion:

After state transformation, the equivalent system is

\[
\begin{align*}
\dot{z}_1 &= \overline{A}_c z_1 + \overline{A}_{12} z_2 + \overline{B}_c u \\
\dot{z}_2 &= \overline{A}_2 z_2
\end{align*}
\]

The input \( u \) has no effect on \( z_2 \). This part of state is uncontrollable. The first sub-system is controllable if \( z_2 = 0 \). If \( z_2 \neq 0 \), then

\[
\begin{align*}
z_1(t_1) &= e^{\overline{A}_c t_1} z_{10} + \int_0^{t_1} e^{\overline{A}_c (\tau - t_1)} \overline{B}_c u(\tau) d\tau + \int_0^{t_1} e^{\overline{A}_c (\tau - t_1)} \overline{A}_{12} z_2(\tau) d\tau \\
z_2(\tau) &= e^{\overline{A}_2 \tau} z_{20}
\end{align*}
\]

Given a desired value for \( z_1 \), say \( z_{1d} \). If we let

\[
\begin{align*}
v(t) &= \int_0^{t_1} e^{\overline{X}_c (\tau - t_1)} \overline{X}_{12} e^{\overline{X}_c \tau} z_{20} d\tau \\
\overline{W}_c(t) &= \int_0^{t_1} e^{\overline{X}_c \tau} \overline{B}_c \overline{B}_c' e^{\overline{X}_c \tau} d\tau \\
u(t) &= -\overline{B}_c' e^{\overline{X}_c t_1 - 0} \overline{W}_c^{-1}(t_1) [e^{\overline{X}_c t_1} z_{10} + v(t_1) - z_{1d}]
\end{align*}
\]

Then you can verify that \( z_1(t_1) = z_{1d} \).
Theorem: Suppose that \( \rho(G^o) = n_1 < n \). Let \( P \) be a nonsingular matrix whose first \( n_1 \) rows are LI rows of \( G^o \). Then

\[
\begin{pmatrix}
\bar{A} \\
\bar{B} \\
\bar{C}
\end{pmatrix} = \begin{pmatrix}
PAP^{-1} \\
PB \\
P^oC^o
\end{pmatrix}
\]

where \( \bar{A}, \bar{B}, \bar{C} \) are in \( \mathbb{R}^{n \times n_1}, \mathbb{R}^{n \times np}, \mathbb{R}^{q \times n_1} \) respectively. Moreover, the pair \( (\bar{A}_o, \bar{C}_o) \) is observable and

\[
\bar{C}_o(sI - \bar{A}_o)^{-1}\bar{B}_o + D = C(sI - A)^{-1}B + D
\]

Discussion: After state transformation, the equivalent system is

\[
\begin{align*}
\dot{z}_1 &= \bar{A}_o \bar{z}_1 + \bar{B}_o u \\
\dot{z}_2 &= \bar{A}_2 \bar{z}_1 + \bar{A}_2 \bar{z}_2 + \bar{B}_o u, \\
y &= \bar{C}_o \bar{z}_1 + Du
\end{align*}
\]

where \( \bar{z}_2 \) may be affected by \( \bar{z}_1 \) but has no effect on \( y \) or \( \bar{z}_1 \).
Summary for today:
• Controllability
• Observability
• Canonical decomposition
  – Controllable/uncontrollable
  – Observable/unobservable

Next Time:
• Controllability and observability continued
  – Controllability/observability decomposition
  – Minimal realization
  – Conditions for Jordan form conditions
  – Parallel results for discrete-time systems
  – Controllability after sampling
• State feedback design (introduction)

Problem Set #9

1. Is the following state equation controllable? observable?
\[
\begin{bmatrix}
0 & 1 & -1 \\
-1 & -1 & 1 \\
-1 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
x \\ x \\ 1
\end{bmatrix}
+
\begin{bmatrix}
0 \\ 1 \\ 0
\end{bmatrix}
u,
\begin{bmatrix}
y \\ 1 \\ 0 \\ 1
\end{bmatrix}x
\]
If not controllable, reduce it to a controllable one;
If not observable, reduce it to an observable one.

2. Is the following state equation controllable? observable?
\[
\begin{bmatrix}
1 & 0 & -2 \\
0 & 1 & -2 \\
0 & 0 & 2
\end{bmatrix}
\begin{bmatrix}
x \\ x \\ 1
\end{bmatrix}
+
\begin{bmatrix}
0 & 1 \\ 0 & 1 \\ 1 & 0
\end{bmatrix}
u,
\begin{bmatrix}
y \\ 1 \\ 0 \\ 1
\end{bmatrix}x
\]
If not controllable, reduce it to a controllable one;
If not observable, reduce it to an observable one.