16.513 Control systems - Last lecture

Last time, we constructed

- Full dimensional estimator
  - SISO case via observable canonical form
  - MIMO case by solving matrix equation

Today: We conclude the design part

- Reduced order observer
- Connection of state-feedback with state estimation
- LQR optimal control
- Rejection of sinusoidal disturbances

Full-Dimensional State Estimators

- The basic idea: make a copy of the original system
  \[ \dot{x} = Ax + Bu; \quad y = Cx \]

- Make correction on \( \dot{x}_e/dt \) based on \( (y - y_e) \)
  \[ \dot{x}_e = Ax_e + Bu + L(y - y_e) \quad y_e = Cx_e \]

- The error dynamics for \( e = x - x_e: \)
  \[ \dot{e} = (A - LC)e \]

- Main issue: designing \( L \) for good convergence of \( e(t) \)
• Main issue: designing L for good convergence of e(t)
\[ \dot{e} = (A - LC)e \]

Under what condition can A-LC be stabilized?
- If (A,C) is observable, then the eigenvalue can be arbitrarily assigned.
- If (A,C) is unobservable, the unobservable subsystem must be stable. Suppose

\[
\begin{bmatrix}
\bar{A} &= PAP^{-1} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_\sigma \end{bmatrix}, \\
\bar{C} &= CP^{-1} = \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix}
\end{bmatrix}
\]

Let \( L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \), then
\[
\begin{bmatrix}
\bar{A} - \bar{L}C &= \begin{bmatrix} \bar{A}_o - L_1\bar{C}_o & 0 \\ \bar{A}_{21} - L_2\bar{C}_o & \bar{A}_\sigma \end{bmatrix}
\end{bmatrix}
\]

The eigenvalues of \( \bar{A}_o - L_1\bar{C}_o \) can be arbitrarily assigned. The eigenvalues of \( \bar{A}_\sigma \) cannot be changed.

For an observable pair (A,C), we studied two approaches to assign the eigenvalues of A-LC
- through observable canonical form
- by solving matrix equation

Next we study:

- Reduced-order estimator
- Combining state estimator with state-feedback
- Summary of feedback design
- Feedback design for discrete-time systems
- LQR optimal control
- Rejection of sinusoidal disturbances
Reduced Dimensional State Estimator

• So far, the dimension of the estimator = n
• Is this really needed especially when q is not small?
  – Assume that \( y = Cx \) with \( C: q \times n, q > 1, C \) full row rank.
• What is the minimum estimator dimension needed?
  – The dimension needed is \( (n - q) \)
• There are two methods
  – By transforming the state equation into a special form: the structure is clear but the procedure is complicated. An earlier method. Will not be covered.
  – By solving matrix equations: simpler procedure.
  
  For MIMO systems, this method offers infinite many solutions. Will be discussed next.

Reduced Dimensional Estimator

• The full-dimensional method via matrix equality can be extended for reduced-dimensional estimator
  – Recall a full dimensional estimator:
    \[
    \begin{align*}
    \dot{x}_e &= Ax_e + Bu + L(y - y_e), \quad y_e = Cx_e \\
    \dot{x}_e &= (A - LC)x_e + Ly + Bu
    \end{align*}
    \]
• A reduced-order equation modified from above:
  \[
  \dot{z} = Fz + Gy + Hu; \quad F \in \mathbb{R}^{(n-q) \times (n-q)}, G \in \mathbb{R}^{(n-q) \times q}, H \in \mathbb{R}^{(n-q) \times p}
  \]
  If \( z(t) \rightarrow Tx \) for some \( T \in \mathbb{R}^{(n-q) \times n} \).
  
  then \[
  \begin{bmatrix}
  y(t) \\
  z(t)
  \end{bmatrix} 
  \rightarrow 
  \begin{bmatrix}
  Cx(t) \\
  Tx(t)
  \end{bmatrix} 
  = 
  \begin{bmatrix}
  C \\
  T
  \end{bmatrix} x(t). \quad \text{If } P := \begin{bmatrix}
  C \\
  T
  \end{bmatrix} \text{ is nonsingular,}
  \]
  
  \[
  P^{-1} \begin{bmatrix}
  y(t) \\
  z(t)
  \end{bmatrix} \rightarrow x(t) \quad \text{State recovered from } y \text{ and } z.\]
The crucial points:

1) Ensure that \( P = \begin{bmatrix} C \\ T \end{bmatrix} \) is nonsingular and
2) \( z(t) \to Tx \)

We first discuss how to ensure 2). Recall

\[
\dot{x} = Ax + Bu \implies T\dot{x} = TAx + TBu
\]

\[
\dot{z} = Fz + Gy + Hu;
\]

Define \( e := Tx - z \)

Then \( \dot{e} = TAx + TBu - Fz - GCx - Hu. \)

If we choose \( T \) such that \( TA = FT + GC \) and \( H = TB \),

Then \( \dot{e} = (FT + GC)x + TBu - Fz - GCx - Hu = F(Tx - z) \)

\[
= Fe
\]

As long as \( F \) is stable, \( e(t) \to 0 \) and \( z(t) \to Tx(t) \).

The algorithm:

– Select \( F \in \mathbb{R}^{(n-q) \times (n-q)} \) having desired estimator eigenvalues which are disjoint from those of \( A \)
– Choose \( G \in \mathbb{R}^{(n-q) \times q} \) such that \( \{F, G\} \) is controllable
– Solve \( TA - FT = GC \) to obtain \( T \in \mathbb{R}^{(n-q) \times n} \)
– If the resulting \( P = \begin{bmatrix} C \\ T \end{bmatrix} \) is non-singular, \( H = TB \) and state estimator can be obtained as

\[
\dot{z} = Fz + Gy + Hu
\]

\[
x_e = P^{-1} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} C \\ T \end{bmatrix}^{-1} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}.
\]

Otherwise, choose a different \( F \) or \( G \) and try again

Note: For randomly chosen \( G \), the probability that \( P \) is nonsingular is 1.
Example (Continued)

\[ \dot{x} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u; \quad y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} x \]

Obtain a reduced-order estimator with pole at -10

- In this case, \( n = 3, q = 2, n-q = 1 \)
- Select \( F ((n-q) \times (n-q)): F = -10 \)
- Select \( G ((n-q) \times q): G = \begin{bmatrix} 1, 0 \end{bmatrix} \sim \{F, G\} \) controllable
- Solve \( TA - FT = GC \) to obtain \( T ((n-q) \times n) \)

\[ T = \begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix} \]

\[ TA = \begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} t_1 + 3t_2 & 2t_1 - t_2 + 2t_3 & t_2 \end{bmatrix} \]

\[ FT = (-10) \cdot \begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix} = \begin{bmatrix} -10t_1 & -10t_2 & -10t_3 \end{bmatrix} \]

\[ GC = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \]

\[ TA - FT - GC = \begin{bmatrix} t_1 + 3t_2 & 2t_1 + 9t_2 + 2t_3 & t_2 + 10t_3 - 1 \end{bmatrix} = 0 \]

\( T \) is obtained as \( T = \begin{bmatrix} 3 & -11 & 93 \\ 454 & 454 & 908 \end{bmatrix} \)

\[ P = \begin{bmatrix} C \\ T \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 3 & -11 & 93 \\ 454 & 454 & 908 \end{bmatrix} \]

\[ H = TB = \begin{bmatrix} 3 & -11 & 93 \\ 454 & 454 & 908 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 83 \\ 908 \end{bmatrix} \]
• Putting things together:

\[ z = Fz + Gy + Hu \]

\[ = -10z + [1 \ 0]y + \frac{83}{908} u \]

\[
x_e = P^{-1} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 1 \ 0 \\ T \end{bmatrix}^{-1} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} -11 & 1 \\ 1 & 93 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 3 & -11 & 1 \\ 454 & 454 & 908 \end{bmatrix}^{-1} \begin{bmatrix} y \\ z \end{bmatrix} \]

---

**Connection of State Estimation and Feedback**

• We will assume controllability and observability
  – How can we use the estimated state?
  – Can we use \( x_e(t) \) in state feedback?
  – What are the complications?

![Block Diagram](image-url)
• How to analyze the combined system?
  – Put all the equations together, and then analyze them
    \[
    \begin{align*}
    \dot{x} &= Ax + Bu; \quad y = Cx \\
    \dot{z} &= Fz + Gy + Hu; \quad TA - FT = GC; \quad x_e = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}
    \end{align*}
    \]
  – Assuming a reduced dimensional estimator
    \[u = r - Kx_e\]
  – There are two equations involving \(x\) and \(z\):
    \[
    \begin{align*}
    \dot{x} &= Ax + B(r - Kx_e) = Ax + B(r - KQ_1y - KQ_2z) \\
    &= (A - BKQ_1C)x - BKQ_2z + Br \\
    \dot{z} &= Fz + Gy + Hu = Fz + GCx + H(r - KQ_1Cx - KQ_2z) \\
    &= (GC - HKQ_1C)x + (F - HKQ_2)z + Hr
    \end{align*}
    \]

  – Introducing the following equivalent transformation:
    \[
    \begin{bmatrix}
    \dot{x} \\
    \dot{z}
    \end{bmatrix} = \begin{bmatrix}
    A - BKQ_1C & -BKQ_2 \\
    GC - HKQ_1C & F - HKQ_2
    \end{bmatrix}
    \begin{bmatrix}
    x \\
    z
    \end{bmatrix} + \begin{bmatrix}
    B \\
    H
    \end{bmatrix} r; \quad y = \begin{bmatrix} C & 0 \end{bmatrix}
    \begin{bmatrix}
    x \\
    z
    \end{bmatrix}
    \]

    \[
    A_L = \begin{bmatrix}
    I & 0 \\
    0 & -T & 1
    \end{bmatrix}, \quad B_L = S, \quad C_L = C_j S^{-1}
    \]

    Transformed estimation error
    \[
    \begin{bmatrix}
    \tilde{x} \\
    \tilde{z}
    \end{bmatrix} = \begin{bmatrix}
    x \\
    z - Tx
    \end{bmatrix}, \quad S, \quad \text{with} \quad \text{S}^{-1} = \begin{bmatrix} 1 & 0 \\
    T & 1
    \end{bmatrix}
    \]

  Recall \[
  \begin{bmatrix}
  Q_1 & Q_2 \\
  C \\
  T
  \end{bmatrix} = Q_1 C + Q_2 T = I, \quad TA - FT = GC; \quad H = TB
  \]

  \[
  \Rightarrow \begin{bmatrix}
  \dot{\tilde{x}} \\
  \dot{\tilde{e}}
  \end{bmatrix} = \begin{bmatrix}
  A - BK & -BKQ_2 \\
  0 & F
  \end{bmatrix}
  \begin{bmatrix}
  \tilde{x} \\
  \tilde{e}
  \end{bmatrix} + \begin{bmatrix}
  B \\
  0
  \end{bmatrix} r; \quad y = \begin{bmatrix} C & 0 \end{bmatrix}
  \begin{bmatrix}
  \tilde{x} \\
  \tilde{e}
  \end{bmatrix}
  \]

  – What can be said about poles of the combined system?
  – They are the union of \(\text{eig}(A-BK)\) and \(\text{eig}(F)\)
– Poles of the combined system: $\text{eig}(A-BK)$ and $\text{eig}(F)$.
– Eigenvalues of state feedback are not affected by the eigenvalues of state estimator $F$, and vice versa.
– Design of state feedback and state estimator can be carried out independently ~ the Separation Property and the Certainty Equivalence Property (not true in general).
– What is the transfer function from $r$ to $y$?
– $e$ is uncontrollable, as it cannot be controlled directly from $r$ or indirectly from $x$ ⇒ Will not show up in $\hat{G}(s)$

\[
\hat{G}(s) = C(sI - A - Bk)^{-1}B
\]

– In deriving the transfer function, initial conditions are assumed to be 0, i.e., $x(0) - x_e(0) = 0$, or $x(0) = x_e(0)$. The dynamics of state estimator therefore will not show up.
– If $x(0) \neq x_e(0)$, the estimation error will show up in $y$. The error will vanish quickly if the eigenvalues of $F$ are further to the left as compared to the eigenvalues of $(A - BK)$.

**Rule of thumb:** The poles of state estimator should be 2 to 3 times faster than the poles of state feedback.
Example. A DC motor driving a load

\[
\begin{array}{c}
\text{u} \\
\hline
\frac{1}{s(s+1)} \\
\hline
\text{y}
\end{array}
\]

Design state feedback with poles at \(-1 \pm j\), and a reduced state estimator with pole at \(-2\).

The two designs can be done separately

**State Feedback:**

**Step 1:** \[\hat{G}(s) = \frac{1}{s(s+1)} = \frac{\beta_1 s + \beta_2}{s^2 + \alpha_1 s + \alpha_2} \sim \alpha_1 = 1, \alpha_2 = 0, \beta_1 = 0, \beta_2 = 1\]

**Step 2:** CCF \[\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u; \quad y = [1 \ 0] x\]

**Step 3:** \[\Delta_d(s) = (s + 1 - j) (s + 1 + j) = s^2 + 2s + 2 \sim \alpha_1 = 2, \alpha_2 = 2\]

**Step 4:** \[\begin{bmatrix} \hat{k}_1 \\ \hat{k}_2 \end{bmatrix} = \alpha_2 - \alpha_1 = 2 - 0 = 2\]
\[\begin{bmatrix} \hat{k}_1 \\ \hat{k}_2 \end{bmatrix} = \alpha_1 - \alpha_1 = 2 - 1 = 1\]
\[\hat{k} = \begin{bmatrix} \hat{k}_1 \\ \hat{k}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\]
\[k = \hat{k}\]

**State Estimator:**

**Step 1:** \(F = -2\)

**Step 2:** Choose \(G = -2\) \(\Rightarrow \{F, G\}\) is controllable

**Step 3:** Solve \(TA - FT = GC\) to obtain \(T\)

\[
\begin{bmatrix} t_1 & t_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 0 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 \\ 2t_1 \end{bmatrix}\]
\[T = \begin{bmatrix} -1 & 1 \end{bmatrix}\]
Step 4: \( P = \begin{bmatrix} C \\ T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \) nonsingular. \( \Rightarrow P^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \).

\( H = TB = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 \)

The estimator:

\[
\dot{z} = Fz + Gy + H u; \quad x_e = P^{-1} \begin{bmatrix} y \\ z \end{bmatrix}
\]

\[
\dot{z} = -2z - 2y + u
\]

\[
x_e = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} y \\ y+z \end{bmatrix}
\]

Combining state feedback and state estimator:

\[
u = r - kx_e = r - \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = r + \begin{bmatrix} -3 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}
\]

– Schematically (with \( \dot{z} = -2z - 2y + u \)):
**Example** (same as in last lecture)

\[ \dot{x} = Ax + bu = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u; \quad y = cx = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} x \]

A full dimensional observer was built as

\[ \dot{x}_e = Ax_e + Bu + L(y - y_e), \quad L = \begin{bmatrix} -3.5 & 24 \\ -3.75 & 15 \\ 3 & 2 \end{bmatrix} \]

So that the estimator poles are at -5, -5, -10

Since A is unstable, the output diverges to infinity.

We need to design a state feedback law. Let the desired eigenvalues of A-bk be -1+j1, -1-j1, -2. The feedback gain is

\[ k=[0.3750 \quad 1.7500 \quad 1.5000] \]
Summary of feedback design

A linear time invariant system

\[ u \xrightarrow{\text{LTI}} y \]

The system can be described by
- a proper rational transfer function (matrix) \( G(s) \)
- state space equation
  \[ \dot{x} = Ax + Bu; \quad y = Cx \]

If by a transfer function, we need to obtain a state-space realization (controllable and observable)

Suppose that \((A,B)\) is controllable and \((A,C)\) is observable.
State feedback gain: find $K$ such that $A-BK$ has the desired eigenvalues;

Observer gain: find $L$ such that $A-LC$ has the desired eigenvalues. Usually assign $\text{eig}(A-LC)$ to be further away from the imaginary axis than $\text{eig}(A-BK)$

When performing simulation, we can break $(A,B,C,0)$ into three components in serial, $B$, $(A,I,I,0)$, $C$ so that we can examine the state $x$
Today’s topics:
- Reduced-order estimator
- Combining state estimator with state-feedback
- Summary of feedback design
- Feedback design for discrete-time systems
- LQR optimal control
- Rejection of sinusoidal disturbances

Feedback design for discrete-time systems

The system:
\[ x[k+1] = Ax[k] + Bu[k], \quad y = Cx[k] \]

- The procedure for designing state feedback and observer is the same as that for continuous-time systems except for the desired eigenvalues for \( A-BK \) and \( A-LC \): eig(\( A-BK \)) and eig(\( A-LC \)) are required to be all inside the unit circle.
- The convergence rate for \( x[k+1] = (A-BK)x[k] \) is faster if the eigenvalues of \( (A-BK) \) have smaller absolute values. \( (A-BK)^k \) goes to 0 faster.
- What happens if the eigenvalues of \( A-BK \) are all 0?
• What happens if the eigenvalues of A-BK are all 0?

\[ (A-BK)^n = 0, \quad (A-BK)^{n+i} = 0, \quad \ldots \]

How to see this? In this case, there exist a similar transformation such that

\[ P(A - BK)P^{-1} = J, \quad J = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_m \end{bmatrix} \]

Each \( J_i \) of the form

\[ J_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \ldots \]

\[ \Rightarrow J_i^n = 0 \quad \Rightarrow J^n = 0 \quad \Rightarrow (A-BK)^n = P^{-1}J^n P = 0 \]

\[ x[k] = (A-BK)^k x[0] = 0 \quad \text{for all } k \geq n. \]

\( u = -Kx \) is called dead-beat control.

Same thing happens for the observer

If A-LC has all zero eigenvalues, we have

\[ e[k] = (A-BK)^k e[0] = 0 \quad \text{for all } k \geq n. \]

\[ x[k] = x_e[k] \quad \text{for all } k \geq n. \]

\( u = Kx_e = Kx, \) same as direct state feedback.

\[ x[n+k] = (A-BK)^k x[n] = 0 \quad \text{for all } k \geq n. \]

\[ x[k] = 0 \quad \text{for all } k \geq 2n. \]

Dead-beat control still achieved.
Today’s topics:

- Reduced-order estimator
- Combining state estimator with state-feedback
- Summary of feedback design
- Feedback design for discrete-time systems
- LQR optimal control
- Rejection of sinusoidal disturbances

LQR optimal control: Motivation

An open-loop system:

\[ \dot{x} = Ax + Bu; \quad y = Cx, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p, \quad y \in \mathbb{R}^q \]

With state feedback \( u = r - Kx \), we have

\[ \dot{x} = (A - BK)x + Br; \quad y = Cx, \]

The closed-loop performance is closely related to the eigenvalues of \( A - BK \), but the relationship can be complicated. Generally, large real parts yield fast convergence rate.

Then why not simply assign eigenvalues with large real parts?

Note that to assign the eigenvalues to the far left of the imaginary axis, the elements of \( K \) have to be large. \( u = Kx \) is large, requiring large control capacity, magnitude, or energy.

- There is a conflict between good response and small control capacity.
Question:
• how can we balance the conflict between good transient response and small control effort?

Problem formulation:
• Use energy of $u$, denoted as $J_1(u)$, to measure control effort:
  – small energy implies small control effort.
• Use the energy of $y$, denoted $J_2(y)$, to measure the quality of the transient response
  – small energy related to fast convergence and small oscillation.
• Construct a performance index as the total sum of energy of the input and the output.
• Add flexibility by using weights, e.g., $J=c_1J_1(u)+c_2J_2(y)$
  – large $c_1$ implies that the control is expensive and we intend to keep them small
  – small $c_1$ indicates that the control is cheap and we don’t care if we need to use large control magnitude or energy.

Linear Quadratic Regulator Problem
• Problem posed and solved by R. E. Kalman (1960)

An LTI system
$$\dot{x} = Ax + Bu; \quad y = Cx, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p, \quad y \in \mathbb{R}^q$$

Assume that $(A,B)$ is controllable and $(A,C)$ observable.

• Objective: Given $Q \in \mathbb{R}^{q \times q}$, $R \in \mathbb{R}^{p \times p}$, $Q \succeq 0$, $R > 0$. 
  For $x(0)=x_0$, find a control $u(t)$, $t>0$, to minimize
  $$J = \int_0^\infty (y'(t)Qy(t) + u'(t)Ru(t))dt$$

$J$ is called the cost function. It measures the total weighted energy of the output and the control.
\[ J = \int_0^\infty (y'(t)Qy(t) + u'(t)Ru(t))dt \]

\( J \) contains two parts:

\[ J_1 = \int_0^\infty y'(t)Qy(t)dt \quad \text{and} \quad J_2 = \int_0^\infty u'(t)Ru(t)dt \]

\( J_1 \) is a measure of energy for the output;
\( J_2 \) is a measure of energy for the input.

Since \( Q \geq 0, R > 0 \), we know \( J_1 \geq 0, J_2 > 0 \).

Usually, \( Q \) and \( R \) are chosen to be diagonal matrices.
Each diagonal element represents a penalty on the corresponding output or input, e.g., suppose \( p=3, q=2 \):

\[
y'Qy = \begin{bmatrix} y_1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = y_1^2 + 5y_2^2
\]

\[
u'Ry = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 10u_1^2 + u_2^2 + 0.1u_3^2
\]

**Problem:**

\[
\min_{u(t)} \int_0^\infty (y'(t)Qy(t) + u'(t)Ru(t))dt \quad \text{t}
\]

s.t. \( \dot{x} = Ax + Bu; \quad y = Cx, \quad x(0) = x_0 \)

**Solution:**

\[ u(t) = -R^{-1}B'Px(t), \quad \text{where} \ P > 0 \text{ satisfies}
\]

\[ PA + A'P - PBR^{-1}B'P + C'QC = 0 \quad (***) \]

**Comments:**

- (***) is called an Algebraic Riccati Equation (ARE)
- Same formula for all initial condition \( x_0 \);
- A simple linear state feedback
- The closed-loop system is

\[ \dot{x} = (A - BR^{-1}BP)x, \quad x(0) = x_0 \]

It is stable.
Proof: From standard calculus we have
\[ \frac{d}{dt}(x'Px) = x'Px' + x'P \dot{x} \]
Also,
\[ \int_{0}^{\infty} \frac{d}{dt}(x'Px) = (x'Px)(\infty) - (x'Px)(0) \quad (2) \]
Combining (1) and (2) to obtain
\[ J - (x'Px)(0) = -(x'Px)(\infty) + \int_{0}^{\infty} \left( y'Qy + u'R + \frac{d}{dt}(x'Px) \right) \, dt \]
\[ = -(x'Px)(\infty) + \int_{0}^{\infty} \left( y'Qy + u'R + x'Px + x'P \dot{x} \right) \, dt \]
Recall that \( P \) satisfies \( PA + A'P - PBR - 1B'P + C'QC = 0 \)
\[ J - (x'Px)(0) = (x'Px)(\infty) + \int_{0}^{\infty} \left( x'B'Px + u'R + u'B'Px + x'P Bu \right) \, dt \]
\[ = (x'Px)(\infty) + \int_{0}^{\infty} (u + x'PBR^{-1})R(u + R^{-1}B Px) \, dt \]
From last slide:
\[ J - (x'Px)(0) = (x'Px)(\infty) + \int_{0}^{\infty} (u + x'PBR^{-1})R(u + R^{-1}B Px) \, dt \]
Now, suppose that \( J \) is finite, we should have \( x(t) \to 0 \) as \( t \) goes to infinity. Thus \( (x'Px)(\infty) = 0 \).
\[ J = (x'Px)(0) + \int_{0}^{\infty} (u + x'PBR^{-1})R(u + R^{-1}B Px) \, dt \]
Since \( R > 0 \), the integrand is nonnegative. To minimize \( J \), we have to choose \( u = -R^{-1}B Px \). By doing this, we also have
\[ \min J = (x'Px)(0) = x(0)'Px(0) \]
Q.E.D.

Comments:
- The optimal cost depends only on \( P \) and \( x(0) \)
- The ARE: \( PA + A'P - PBR - 1B'P + C'QC = 0 \) has many solutions. But there is only one \( P > 0 \).
- The optimal control is a linear state feedback.
- The closed-loop matrix \( A - BR^{-1}B'P \) is stable.
**Example** (same as in last lecture)

\[
\dot{x} = Ax + bu = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u; \quad y = Cx = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} x
\]

Case 1: Pick \( Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 1, \quad (Q = [1 0; 0 1]) \)

\[
k = \text{lqr}(A,b,C'*Q*C,R), \quad k = [1.3260 \quad 2.1651 \quad 2.3134] \\
\text{eig}(A-b*k) = \{-0.3588, -3.3858 + 0.1672i, -3.3858 - 0.1672i\}
\]

Case 2: \( Q = [10 \ 0; 0 \ 10]; \quad R = 1, \)

\[
k = [1.0197 \quad 4.0322 \quad 5.2158], \\
\text{eig}(A-b*k) = \{-0.4702, \ -3.2555, \ -7.5618\}
\]

Case 3: \( Q = [1000 \ 0; 0 \ 1000]; \quad R = 1, \)

\[
k = [-1.8875 \quad 31.6460 \quad 46.6354] \\
\text{eig}(A-b*K) = \{-0.4961, \ -3.2487, \ -70.7616\}
\]

- 0.4963 as \( Q \) goes to infinity

Control \( u \): larger \( Q \) results in larger magnitude of \( u \). Larger \( Q \), heavier weight on \( y \), control is relatively cheaper.

![Graph](image-url)

**Legend**
- **Red**: \( Q = I, \ R = 1; \)
- **Blue**: \( Q = 10I, \ R = 1; \)
- **Green**: \( Q = 1000I, \ R = 1. \)
Output $y_1$: Larger $Q$ results in faster convergence rate of $y$ and smaller magnitude

Red: $Q=I$, $R=1$;  
Blue: $Q=10I$, $R=1$;  
Green: $Q=1000I$, $R=1$.

Output $y_2$: Larger $Q$ results in faster convergence rate of $y$ and smaller magnitude

Red: $Q=I$, $R=1$;  
Blue: $Q=10I$, $R=1$;  
Green: $Q=1000I$, $R=1$. 
Simulation with feedback from estimated state
Control \( u \): larger \( Q \) results in larger magnitude of \( u \)

Output \( y_1 \): Larger \( Q \) results in faster convergence rate of \( y \) and smaller overshoot
Output $y_2$: Larger $Q$ results in faster convergence rate of $y$ and small overshoot.

![Graph showing responses under different weight on $y_2$ (same weight on $y_1$)](image)

Responses under different weight on $y_2$ (same weight on $y_1$)

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 1$$  
$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}, \quad R = 1$$  
$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1000 \end{bmatrix}, \quad R = 1$$

- Red: $Q=I, R=1$;
- Blue: $Q=10I, R=1$;
- Green: $Q=1000I, R=1$.

Blue: $y_1$

Green: $y_2$
Today’s topics:
- Reduced-order estimator
- Combining state estimator with state-feedback
- Summary of feedback design
- Feedback design for discrete-time systems
- LQR optimal control
- Rejection of sinusoidal disturbances

Rejection of sinusoidal disturbances

The system
\[
\dot{x} = Ax + Bu + BG \, d, \quad y = Cx
\]

The disturbance \(d\) is a sinusoidal signal with frequency \(\omega\),
\[d(t) = d_m \sin(\omega t + \theta)\]. If we know exactly the magnitude
and the phase of \(d(t)\), then we can let \(u = -Kx - Gd\), then
\[
\dot{x} = Ax - BKx - BGd + BGd = (A - BK)x
\]

If \((A - BK)\) is stable, then \(x(t) \to 0\).

**Question:** What can we do if the magnitude and phase
of \(d(t)\) is unknown?

**Solution:** build an observer.
The key point: we can represent \( d(t) = d_m \sin(\omega t + \theta) \) as the output of a linear system:

\[
\dot{v} = Sv = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} v, \quad d = c_v v = [1 \ 0] v
\]

Recall that \( e^{st} = \begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix} \)

\[
d(t) = c_v e^{st} v(0) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix} \begin{bmatrix} v_{10} \\ v_{20} \end{bmatrix}
\]

Since \( (v_{10}, v_{20}) = (\rho \sin \theta, \rho \cos \theta) \), for \( \rho = || v_0 || \), some \( \theta \in [0, 2\pi) \),

\[
d(t) = \rho (\sin \theta \cos \omega t + \cos \theta \sin \omega t)
\]

\[= \rho \sin(\omega t + \theta)\]

Hence \( d_m = || v_0 || \) and \( \theta \) are uniquely determined by the initial condition of \( v \) and there is a one to one corresp.

On the other hand, given \( d_m \) and \( \theta \), \( v(0) = [d_m \sin(\theta) \ d_m \cos(\theta)]' \)

The magnitude and the phase are the polar coordinate of \( v(0) \).

Now we have

\[
\dot{x} = Ax + Bu + BGc_v v, \quad y = Cx
\]

\[
\dot{v} = Sv
\]

Let \( z = \begin{bmatrix} x \\ v \end{bmatrix} \)

\[
\dot{z} = \begin{bmatrix} A & BGc_v \\ 0 & S \end{bmatrix} z + \begin{bmatrix} B \\ 0 \end{bmatrix} u, \quad y = [C \ 0] z
\]

\[\Rightarrow \dot{z} = A_z z + B_z u, \quad y = C_z z\]

If \( (A_z, C_z) \) is observable, then an observer (with state \( z_e \)) can be constructed to estimate the state \( z \). Partition \( z_e \) as

\[
z_e = \begin{bmatrix} x_e \\ v_e \end{bmatrix}
\]

\[\Rightarrow x_e \rightarrow x, \quad v_e \rightarrow v, \quad c_v v_e(t) \rightarrow c_v v(t) = d(t) = d_m \sin(\omega t + \theta)\]

The disturbance \( d(t) \) is reconstructed as \( d_e(t) = c_v v_e(t) \).
Now we have $x$ and $v$ estimated with $x_e$ and $v_e$, let

$$u(t) = -Kx_e(t) - Gc_vv_e(t)$$

The closed loop system is:

$$\dot{x} = Ax - BKx_e - BGc_vv_e + BGd$$

$$= (A - BK)x + BK(x - x_e) + BG(c_vv - c_vv_e)$$

Since $x - x_e \to 0$, $c_vv - c_vv_e \to 0$ and $A - BK$ is stable, we have $x \to 0$.

Again, the key point is to consider the disturbance as part of the original system and is estimated with an observer.

The procedure of design is illustrated in the following example.

**Example:**

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u + d), \quad d(t) = d_m \sin(t + \theta), \quad y = [1 \ 0]x$$

Since $\omega = 1$, take $S = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $c_v = [1 \ 0]$

Then $\dot{v} = Sv$, $d(t) = c_vv(t)$ Let $z = \begin{bmatrix} x \\ v \end{bmatrix}$, we have

$$\dot{z} = A_zz + B_zu, \quad y = C_zz$$

where $A_z = \begin{bmatrix} A & Bc_v \\ 0 & S \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, $B_z = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $C_z = [1 \ 0 \ 0 \ 0]$

Construct the observer,

$$\dot{z}_e = A_zz_e + B_zu + L(y - C_zz_e),$$

Need to design $L$.
The observer:
\[ \dot{z}_e = A_z z_e + B_z u + L(y - C_z z_e), \]

The desired eigenvalue for the observer: -4+j4, -4-j4, -6, -10.
The resulting L is
\[ L = \begin{bmatrix} 25 & 245 & 968 & 1701 \end{bmatrix}'; \]

Next design the state feed back gain K.
The desired eigenvalue for A-BK: -1+j1, -1-j1
The resulting K is, \( K = \begin{bmatrix} 3 & 3 \end{bmatrix} \)
The control law:
\[ u = -Kx_c - cv_c = -[K cv]z_e = -[3 3 1 0]z_e \]
• The disturbance rejection problem mentioned above is an output regulation problem.
• The method can be extended to deal with the case where \( d(t) \) has several frequency components, such as
  \[
  d(t) = d_1 \sin(\omega_1 t + \theta_1) + d_2 \sin(\omega_2 t + \theta_2) + \ldots
  \]
or a periodic signal with a few harmonics
• The method can also be extended for the purpose of tracking a sinusoidal or periodic signals.
• One of my recent paper studies output regulation with input constraints


---

Project and Final Exam: Due 4pm, Dec 14, 2014

Final exam problems will be sent to your email box at uml. at 9am, Dec 13 (Saturday).

The written part of the project should be complete with all the results clearly presented. 3 points out of 25 will be given on presentation.

All the Matlab and Simulink files for the project and the final exam should be contained in a CD for possible verification.

The project and final exam should be done independently.
Problem Set #12

Problem 1: The open-loop system
\[
\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 4 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.
\]

1) Assume that \( x \) is available for state feedback. Design an LQR control law by letting \( R=1 \) and choosing \( Q \) so that all the elements of the feedback gain \( K \) have absolute value less than 50. Requirement: \( |y_1(t)|, |y_2(t)| \leq 0.05 \) for all \( t > 5 \). Plot \( y_1(t) \) and \( y_2(t) \) in the same figure for \( t \in [0,15] \).

2) Assume that only the output \( y \) is available. Design an observer so that the poles of the observer are -5+j5, -5-j5, -10. Choose the observer gain so that all the elements have absolute value less than 80. Form a closed-loop system along with the LQR controller in part 1). Plot \( y_1(t) \) and \( y_2(t) \) in the same figure for \( t \in [0,15] \).

Problem 2: The open-loop system
\[
\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 4 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} (u + d), \quad y = [1 \quad 1 \quad 1] x, \quad x(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
\]

The disturbance is \( d(t) = d_m \sin(2t + \theta) \).

Construct a feedback law from \( u \) and \( y \) such that the disturbance is rejected. Given the initial condition of \( x \) and \( d(t) = \sin(2t) \). Adjust controller parameters (\( K \) and \( L \)) such that \( |u(t)| \leq 20 \) and \( |y(t)| \leq y_{\text{max}} \) for all \( t \) and \( y_{\text{max}} \) is as small as possible. Plot \( u(t) \) in one figure, \( y(t) \) in another figure.

Follow the steps of the example on slide 52. Be careful with the dimension of the matrices.

Due together with your project and final exam.