The Impact of Process Deterioration on Production and Maintenance Policies

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Abstract

This paper examines a single-stage production system that deteriorates with production actions, and improves with maintenance. The condition of the process can be in any of several discrete states, and transitions from state to state follow a semi-Markov process. The firm can produce multiple products, which differ by profit earned, expected processing time, and impact on equipment deterioration. The firm also has the ability to perform different maintenance actions, which differ by cost incurred, expected down time, and impact on the improvement of the process condition. The firm needs to determine the optimal production and maintenance choices in each state in a way that maximizes the long-run expected average reward per unit time.

The paper makes four sets of contributions: 1) It introduces three critical ratios for the firm’s optimal choices. The first enables the firm to decide whether to manufacture or maintain the equipment, the second allows the firm to choose the product to be manufactured, and the third determines the best maintenance action. The economic interpretations of these three critical ratios provide managerial insights. 2) The paper shows how these three critical ratios can be combined in order to determine the optimal policy, simultaneously accounting for the trade-offs involving production profits, maintenance costs, and the impact of these actions on the process condition. We show how these results generalize to problem settings with an arbitrary number of machine states. 3) The paper demonstrates the impact of market demand conditions on the choice of the optimal policy and the critical ratios. And, 4) it develops a set of sufficient conditions that lead to monotone optimal policies. These conditions generalize those reported in earlier studies.
1 Introduction

In many manufacturing environments, the condition of the process or equipment has a significant impact on the quantity and quality of units produced. Consider the case of semiconductor manufacturing in which a chip maker must decide how to allocate production resources among leading-edge and lagging-edge technology products. High-technology products earn a greater profit than low-technology products, but they are also more complex and thus take more time to produce. This increase in production time causes greater deterioration of the manufacturing process, which, in turn, increases the likelihood of quality problems. The chip maker also has the option of performing maintenance, which returns the process to an improved state. Here, too, there is more than one option. At one end of the spectrum, major improvement in the process condition can be achieved by performing a lengthy and costly maintenance procedure; thus, a major maintenance action has a greater likelihood of improving the process condition. At the other end of the spectrum, a minor maintenance procedure can be performed which will cost less and take less time, but has a smaller probability of returning the process to an improved state. Thus, operating this type of system over time requires a manager to answer a series of interconnected questions:

1. Whether to manufacture a product (which may result in the deterioration of the process) or maintain the equipment for a possible improvement;
2. If the decision is to manufacture a product, then which product to manufacture as the choice influences the deterioration of the process differently;
3. If the decision is to maintain the equipment, then which maintenance action to implement as the choice influences the improvement of the process differently.

This paper presents a semi-Markov decision process model to explore the trade-offs involved in answering these three questions. The objective of the model is to determine a course of action that will maximize the long-run expected average reward.

While there has been much research on production systems with deteriorating process condition, our inclusion of multiple products and multiple maintenance actions, as well as our approach, sets this work apart from the majority of previous research in this area. After developing initial insights about the structural properties of the problem in a four-state setting, we show how these lessons can be extended to a problem with any number of states. We integrate market demand conditions by enforcing a minimum production requirement for each product, and we explore how this change affects the critical ratios and the resulting optimal policy.

The paper makes four contributions. First, it develops three types of critical ratios which allow the comparison of any two actions in a given state. Following the three questions above, one critical ratio determines whether the firm should produce a product or perform maintenance, another determines which product is optimal in states where production is preferred, and a third critical ratio identifies which maintenance action
is optimal in states where maintenance is preferred. The first critical ratio can be interpreted as the \textit{reservation price}, i.e., the minimum amount of money the decision maker should earn in order to justify production over maintenance. Similarly, the second critical ratio represents the minimum amount of profit the firm needs to earn to switch from a low-end product to a high-end product. The third critical ratio establishes an upper bound on the maximum amount of money that the decision maker should be willing to spend on major maintenance. Second, the paper demonstrates how the critical ratios can be combined to determine the optimal action in a particular machine state given all of the possible alternatives. Their combination enables the decision maker to \textit{simultaneously} account for the trade-offs involving profit benefits versus deterioration probability and cost versus improvement probability. Third, the paper shows the influence of minimum throughput requirements on the choice of the optimal policy and the critical ratios. It proves that the frequency and timing of maintenance play a strategic role in increasing the throughput of a high demand product. Fourth, the paper develops a set of conditions which are sufficient to ensure that a monotone policy is optimal. In monotone policies, the firm manufactures the high-end product in better states, and switches to the low-technology product as the process deteriorates. Minor maintenance is employed as the process continues to deteriorate, eventually employing major maintenance at significant deterioration levels. The conditions that lead to monotone policies are much more general than those reported in previous research. We demonstrate the utility of the new conditions by presenting examples that do not meet the previously reported conditions but that still have monotone optimal policies.

The paper proceeds as follows. The next section presents an overview of the relevant literature. The basic model is developed in Section 3. Section 4 presents several generalizations of the model and discusses how our results go beyond those previously reported. Conclusions and managerial insights are in Section 5. All proofs and technical derivations are provided in the Appendix.

2 Literature Review

Many researchers have studied problems at the intersection of production and maintenance scheduling, i.e., where the state of the equipment affects the production process in some way. Production systems with variable yield have received much attention, as discussed in the extensive review by Yano and Lee (1995). Much of the work in this area, starting with Rosenblatt and Lee (1986) and Porteus (1986), has been a variation of the economic manufacturing quantity (EMQ) model. The central questions is: How much of a product should be produced given that some fraction of it may be defective? The process begins in an “in-control” state but may shift to an “out-of-control” state, which results in defective products. Groenevelt \textit{et al.} (1992a), El-Ferik (2008), and Groenevelt \textit{et al.} (1992b) show that the optimal batch sizes are bigger when the possibility of equipment failure is incorporated. These models account for the risk of unknowingly producing defective items, and the equipment state affects only the quantity of production, but not the quality.
These early works have been extended in many ways. Hariga and Ben-Daya (1998) relax some of the assumptions about the equipment’s shift to the out-of-control state and develop structural properties for this more general case. Lee and Rosenblatt (1989) and Lee and Park (1991) investigate different cost structures that depend on when defective items are detected. Lee and Rosenblatt (1987), Porteus (1990), Makis (1998), and Kim et al. (2001) incorporate inspections into the decision model, allowing early detection of the out-of-control state. Boone et al. (2000) extend the model to include machine failures, and Makis and Fung (1998) include both inspections and machine failures. The model proposed by Ben-Daya (2002) allows for imperfect maintenance, i.e., preventive maintenance that may not return the process to the in-control state.

Departing from the EMQ approach, Gilbert and Emmons (1995) develop a model of a job shop in which defective items must be reworked and reduce the production capacity. Inspections reveal if the process is out of control, and a restoration action returns the process to the in-control state. The objective is to determine an inspection and restoration policy that maximizes throughput. Gilbert and Bar (1999) extend these ideas to a small batch production system where they show that a control limit policy is optimal, suggesting that it is ideal to restore the equipment condition when the number of units remaining in a batch exceed a certain threshold.

Sloan (2004) models a system with multiple machine states, where the output follows a binomial distribution that depends on the equipment state. Iravani and Duenyas (2002) construct an integrated production and maintenance model in which the decisions at each epoch are restricted to: produce one unit (rather than in batches), perform maintenance, or do nothing.

While the papers mentioned above consider single-product systems, a significant amount of research has investigated multi-product systems. For example, Lee (2004) examines a traditional job-shop scheduling problem in the context of unreliable equipment. Cassady and Kutanoglu (2005) extend this type of work by simultaneously determining the maintenance and production schedules. Aghezzaf et al. (2007) also aim to combine production and maintenance scheduling, this time in the context of a multi-product, batch production system with failure-prone equipment. In all of these papers as well, however, the state of the process is limited to either “up” or “down.” In the “down” state, no production is possible; in the “up” state, all output is of perfect quality.

Sloan and Shanthikumar (2000, 2002) study multi-product systems with deteriorating process condition in which the process state can be influenced by the decision maker and where the state affects the yield of each product differently. However, both studies assume that all products have the same processing times and the machine state transitions are independent of the product manufactured. Kazaz and Sloan (2008) consider a single-stage system in which processing times and machine state transition probabilities both vary by product type. Conditions are developed that define the exact optimality point for each product and state; however, no demand requirements are considered. In addition, in all of these papers only one maintenance action is
allowed, and this action returns the process to the best state with probability one.

In most situations, there are maintenance actions short of total replacement that can be taken to reduce or alter the rate of process deterioration. Wang (2002) provides an extensive review of the maintenance literature. The works that relate most closely to the current problem include single-machine systems with Markov deterioration and multiple maintenance actions. Such models have been formulated in the context of completely observable state information (Hopp and Wu, 1990), partially observable state information (Hopp and Wu, 1988), and imperfect maintenance (Su et al., 2000). None of these models, however, explicitly accounts for the impact of equipment condition on the production process.

In short, there has been relatively little work on systems that have the following characteristics: multiple products are produced, the quality of output depends on the equipment or process state, the process state can be influenced by the decision maker, and multiple “maintenance” actions are allowed. One model that addresses all of these issues — and therefore most closely relates to ours — is that of Sloan (2008), which studies a multi-product manufacturing system in which multiple maintenance actions are available. The processing times and associated machine state transition probabilities both depend on the type of production and maintenance actions being employed. Sufficient conditions are developed that ensure a monotone policy with respect to both production and maintenance actions. Our paper also provides sufficient conditions; however, the ones presented here are significantly more general than those presented in Sloan (2008). We demonstrate the utility of the new conditions by presenting example problems with monotone optimal policies that do not meet the conditions of Sloan (2008) but do meet the new set of conditions.

3 The Model

This section presents a model to determine a firm’s production and maintenance decisions in a single-stage manufacturing process. The equipment used in the process is described by a discrete number of states denoted by \( i = 1, \ldots, N \). As the equipment condition deteriorates, state \( i \) moves from 1 (best state) to \( N \) (worst state). The equipment condition deteriorates as production takes place and improves with maintenance. The firm is capable of producing multiple products, where each product influences the deterioration process differently. We denote \( P_1 \) as a standard, low-end technology product and \( P_2 \) as a new, high-end technology product. Similarly, the firm can take various maintenance actions that result in varying improvements in the state of the equipment. We denote \( M_1 \) as a minor maintenance action and \( M_2 \) as a major maintenance action. We define the set of production decisions as \( P = \{ P_1, P_2 \} \) and the set of maintenance decisions as \( M = \{ M_1, M_2 \} \).

The firm’s objective is to determine a course of action that maximizes the long-run expected average reward. As a result, the manager is faced with the following three decisions at each decision epoch: 1) Whether to manufacture a product or perform maintenance, 2) If production is picked, then which product to produce, and 3) If maintenance is picked, then which type of maintenance to perform.
Each of the above three decisions has trade-offs for the manufacturer. In the case of the first decision, the firm has to choose between manufacturing and maintenance actions. When manufacturing is the choice, the firm earns a profit via its production but risks the deterioration of the equipment further. However, when maintenance is the choice, the firm incurs a cost for maintaining the system (rather than earning profit) but increases the likelihood of improving the equipment condition. In addition, more time spent maintaining the equipment means less time producing, so while the improved equipment condition will increase the yield, the net throughput may actually decrease.

We define the total profit generated in state \( i \) as the amount of yield for product \( i \). In order to reflect the operating environment of a manufacturer, we require that the firm manufactures in the best state, i.e., \( a_1 \in \{ P1, P2 \} \), and that it performs maintenance in the worst state, i.e., \( a_N \in \{ M1, M2 \} \).

State transition probabilities depend on the choices of manufacturing and maintenance actions. We define \( p_{ij}^{a} \) as the probability that the machine moves from state \( i \) to \( j \) when action \( a \) is taken in state \( i \). When a manufacturing action is taken in state \( i \), the equipment either stays in its current state or deteriorates to a worse state, but cannot improve to a better state. In other words, \( p_{ij}^{a} > 0 \) for all \( i \leq j \) where \( i = 1, \ldots, N-1 \) and \( j = i, \ldots, N \), and \( p_{ij}^{a} = 0 \) for all \( i > j \) where \( i = 1, \ldots, N-1 \) and \( j = 1, \ldots, i-1 \). On the other hand, when a maintenance action is taken in a state \( i \), the equipment either stays in its current state or improves to a better state, but cannot deteriorate to a worse state. Thus, \( p_{ij}^{a} > 0 \) for all \( i \geq j \) where \( i = 2, \ldots, N \) and \( j = 1, \ldots, i \), and \( p_{ij}^{a} = 0 \) for all \( i < j \) where \( i = 1, \ldots, N-1 \) and \( j = i+1, \ldots, N \). For any state \( i \) where \( 1 < i < N \), the machine state transition probabilities can be summarized as follows:

\[
p_{ij}^{a} \begin{cases} 
= 0 & \text{when } j < i; \\
> 0 & \text{when } j > i \quad \text{for } a = P1, P2 \quad \text{where } p_{ij}^{P1} < p_{ij}^{P2}; \\
> 0 & \text{when } j = i; \quad \text{and } p_{ii}^{P1} < p_{ii}^{P2}; \quad \text{and } p_{ii}^{M1} > p_{ii}^{M2}; \\
= 0 & \text{when } j > i; \quad > 0 & \text{when } j < i \quad \text{for } a = M1, M2 \quad \text{where } p_{ij}^{M1} < p_{ij}^{M2}. 
\end{cases}
\]  

(1)

Regarding the choice of product to be manufactured, the firm has to consider another trade-off as well. In this case, the firm needs to decide whether to earn a regular profit with a lower risk of deterioration versus a higher profit that comes with an increased likelihood of deterioration. The profit earned from each product is denoted as \( \rho_{ai} \), where \( a_i \in P \). As consumers are willing to pay more for a new technology item and less for a standard product, we assume that the standard product earns a smaller profit than the new product; i.e. \( \rho_{P1} < \rho_{P2} \). The yield from manufacturing activities also vary by product and by state. We define \( y_{i,ai} \) as the amount of yield for product \( a_i \) when manufactured in state \( i \), and assume that \( y_{i,ai} \) is decreasing in state \( i \) as the firm obtains a lower number of non-defective products with deteriorating process conditions. We define the total profit generated in state \( i \) by production action \( a_i \) as \( r_{i,ai} = \rho_{ai} y_{i,ai} \). In a typical operating environment for a semiconductor manufacturer, the high-end product generates a larger total profit.
in each state, i.e., \( r_{i,P1} < r_{i,P2} \) in each state \( i = 1, \ldots, N - 1 \). However, the processing times also vary by product and by state, and can make the manufacturing of the high-end product less desirable. We define the expected processing time for these two production choices in a state \( i \) as \( \tau_{i,P1} \) and \( \tau_{i,P2} \), respectively. In semiconductor manufacturing, new products typically require a higher circuit density and have a longer expected processing time than the older products. Reflecting this fact, we assume \( \tau_{i,P2} > \tau_{i,P1} \) in each state \( i = 1, \ldots, N - 1 \). The consequence of a longer expected processing time is that the equipment is more likely to deteriorate when product \( P2 \) is manufactured. Thus, it is appropriate to define the state transition probabilities as \( p_{ii}^{P1} > p_{ii}^{P2} \) corresponding to the fact that the equipment would stay in its current state with a higher probability when product \( P1 \) is produced than when product \( P2 \) is produced. Alternatively for state \( i \), the firm has \( \sum_{j=i+1}^{N} p_{ij}^{P1} = (1 - p_{ii}^{P1}) \) and \( \sum_{j=i+1}^{N} p_{ij}^{P2} = (1 - p_{ii}^{P2}) \), and the sum of deterioration probabilities is lower when product \( P1 \) is produced than when product \( P2 \) is manufactured. It should be emphasized here that the decreasing behavior of \( r_{i,a_i \in P} \) and the increasing behavior of \( \tau_{i,a_i \in P} \) are not necessary in developing our results in Section 3. However, they represent the operating environment for semiconductor manufacturers, and more importantly, are useful in explaining the structural results regarding monotone optimal policies in Section 4.

The third question considers the trade-off in alternative maintenance actions. The standard maintenance action \( M1 \) has a cost of \( c_{i,M1} > 0 \) and its expected processing time in state \( i \) is defined as \( \tau_{i,M1} \). In this case, the firm can take a more involved maintenance action described by \( M2 \). The cost of maintenance action \( M2 \) is higher than that of \( M1 \): \( c_{i,M2} > c_{i,M1} \) for all states \( i = 2, \ldots, N \). However, the likelihood of improving the process condition through \( M2 \) is also higher. Thus, the firm has \( p_{ii}^{M2} < p_{ii}^{M1} \) and the sum of improvement probabilities are \( \sum_{j=1}^{i-1} p_{ij}^{M2} = (1 - p_{ii}^{M2}) > \sum_{j=1}^{i-1} p_{ij}^{M1} = (1 - p_{ii}^{M1}) \). It is assumed that the major maintenance action \( M2 \) has a longer expected processing time than the minor maintenance action \( M1 \) in each state, and therefore \( \tau_{i,M2} > \tau_{i,M1} \) in each state \( i = 2, \ldots, N \).

Process deterioration can also influence the cost of maintenance. For example, as the equipment deteriorates more, the firm might have to spend more effort and money on maintenance. Therefore, \( c_{i,a_i \in M} \) and \( \tau_{i,a_i \in M} \) can be considered as increasing in \( i \). Once again, the increasing behavior of \( c_{i,a_i \in M} \) and \( \tau_{i,a_i \in M} \) are not necessary for the results developed in Section 3. However, they prove to be useful in explaining the conditions that lead to monotone optimal policies in Section 4.

Note that the state transition probabilities for maintenance actions are defined in a more general way than in most previous research. For example, most research assumes that maintenance returns the machine to the best state with probability one; we make no such assumption. In addition, the transition probability from state \( i \) to \( j \) where \( i > j \) when maintenance action \( a_i \in M \) is taken in state \( i \) need not be equal to the transition probability from state \( k \) to \( j \) where \( k > j \) when the same maintenance action is taken in state \( k \neq i \). Nor does the paper make an assumption such as \( p_{ij}^{a_i} = p_{kj}^{a_k} \) where \( \{i, k\} > j \) for each maintenance action \( a_i \in M \).
Moreover, this paper does not assume that improvement probabilities with a constant number of states are equal. For example, $p_{ij}^{a_i}$ and $p_{i+1,j+1}^{a_i}$ where $i > j$ are not necessarily equal for a maintenance action $a_i \in \mathbf{M}$.

It should be observed that the time between decisions epochs, the state transition probabilities, the profits and the maintenance costs are dependent only on the action taken in the current state. Therefore, the problem can be modeled as a Semi-Markov Decision Process (SMDP). A time-invariant (or, stationary) policy results in a discrete-time Markov chain that represents the machine condition at decision epochs, and is referred to as the Embedded Markov Chain (EMC). The state transition probabilities in this problem characterize the evolution of the EMC over time as they can be defined as $p_{ij}^{a_i} = \Pr \{ X_{t+1} = j \mid X_t = i, a_t = a \}$ where $X_t$ denotes the machine state and $a_t$ describes the action taken at decision epoch $t$. While there are several approaches to solving this type of problem (interested readers can review Puterman (1994)), we utilize a policy improvement approach. In this approach, we begin with a reference policy and compare it to other policies that differ in its actions in various states. The policy that maximizes the long-run average expected value is referred to as the optimal policy.

We define $A = [a_i \mid i = 1, \ldots, N]$ as a stationary policy that describes the firm’s action $a_i$ in state $i$ and $\Pi_i (A)$ corresponds to the steady-state probability that the EMC is in state $i$. It should be observed that given the definition of the state transition probabilities, the EMC induced by a stationary policy $A$ has a single closed set of recurrent states (i.e., is unichain). The implication of having a single set of recurrent states is that, regardless of the initial state of the equipment, there exists a unique set of steady-state probabilities. However, the steady-state probability, defined as $\Pi_i (A)$ for state $i$, depends on the actions taken in all states. Because the profits and costs depend only on the actions taken in the current state, $EV (A) = \sum_{i=1}^{N} (1_{a_i \in \mathbf{P}} r_{i,a} \Pi_i (A) - 1_{a_i \in \mathbf{M}} c_{i,a} \Pi_i (A)) / \sum_{i=1}^{N} (\tau_{i,a} \Pi_i (A))$ is the average reward rate of policy $A$, where $1_{a_i \in \mathbf{P}}$ is the indicator whether a production action is taken in state $i$ and $1_{a_i \in \mathbf{M}}$ indicates whether a maintenance action is taken. A policy is the optimal policy, described as $A^*$, when $EV (A^*) \geq EV (A)$ for all stationary policies $A$. The optimal action in state $i$ is defined as $a_i^*$, and it can be shown that the optimal policy specifies only one action per state (Puterman, 1994).

The problem variant with four machine states ($i = 1, 2, 3, 4$), two products ($P_1, P_2$), and two maintenance actions ($M_1, M_2$) is sufficient to develop the insight necessary for the structural properties. While Section 3 analyzes the problem with four states, its results are generalized by considering an arbitrary number of states in Section 4. In the four-state variant of the problem, the firm manufactures in the best state, i.e., $a_1 \in \mathbf{P}$, and performs maintenance in the worst state, i.e., $a_4 \in \mathbf{M}$. In the intermediate states ($i = 2, 3$), the firm has to determine an answer to all three questions described earlier: 1) whether to manufacture a product or maintain the equipment, 2) if manufacturing is preferred, then, which product to produce, and 3) if maintenance is preferred, then whether to employ a minor or major maintenance action; thus, $(a_2, a_3) \in \{ P_1, P_2, M_1, M_2 \}$. This results in four groups of policies that feature production and maintenance actions, described with $P$.
and $M$, respectively. These policies are classified as Group 1: $[P,P,P,M]$, Group 2: $[P,P,M,M]$, Group 3: $[P,M,P,M]$, and Group 4: $[P,M,M,M]$. As a result, the firm has a total of 64 policies:

Table 1: Comprehensive list of policies in a four-state problem

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<td>$A_{64} = [P2, M2, M2, M2]$</td>
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</tbody>
</table>

A comparison of the steady-state probabilities in the four groups of policies provides useful observations.

We express the steady-state probability in state $i$ as $\Pi_i (A_n) = \hat{\Pi}_i (A_n) / \sum_{i=1}^N \hat{\Pi}_i (A_n)$, where $\hat{\Pi}_i (A_n)$ is the numerator term for the steady-state expression of state $i$ for policy $A_n$. The $\pi_i (A_n)$ values for the 64 policies above can be expressed as

$$\hat{\Pi}_1 (A_n) = \begin{cases} 
(1 - p_{22}^a) (1 - p_{33}^a) p_{44}^a & \text{for } n = 1, \ldots, 16; \\
(1 - p_{22}^a) p_{31}^a (p_{41}^a + p_{42}^a) + p_{22}^a p_{31}^a p_{42}^a + p_{22}^a p_{43}^a & \text{for } n = 17, \ldots, 32; \\
(1 - p_{22}^a) (1 - p_{33}^a) (p_{41}^a + p_{43}^a) & \text{for } n = 33, \ldots, 48; \\
(1 - p_{22}^a) (1 - p_{33}^a) (1 - p_{44}^a) & \text{for } n = 49, \ldots, 64. 
\end{cases}$$

$$\hat{\Pi}_2 (A_n) = \begin{cases} 
(1 - p_{33}^a) [p_{12}^a p_{44}^a + (1 - p_{11}^a) p_{42}^a] & \text{for } n = 1, \ldots, 16 \text{ and } n = 33, \ldots, 48; \\
(1 - p_{33}^a) [p_{12}^a (1 - p_{44}^a) + p_{14}^a p_{42}^a] + p_{32} [p_{13}^a (1 - p_{44}^a) + p_{14}^a p_{43}^a] & \text{for } n = 17, \ldots, 32; n = 49, \ldots, 64. 
\end{cases}$$

$$\hat{\Pi}_3 (A_n) = \begin{cases} 
p_{13}^a (1 - p_{22}^a) p_{41}^a + p_{22}^a [p_{12}^a p_{41}^a + (1 - p_{11}^a) p_{42}^a] + (1 - p_{11}^a) (1 - p_{22}^a) p_{44}^a & \text{for } n = 1, \ldots, 32; \\
(1 - p_{22}^a) [p_{13}^a (1 - p_{44}^a) + p_{14}^a p_{43}^a] & \text{for } n = 33, \ldots, 64. 
\end{cases}$$

$$\hat{\Pi}_4 (A_n) = \begin{cases} 
(1 - p_{11}^a) (1 - p_{22}^a) (1 - p_{33}^a) & \text{for } n = 1, \ldots, 16; \\
p_{14}^a (1 - p_{22}^a) p_{31}^a + p_{24} [p_{12}^a p_{31}^a + (1 - p_{11}^a) p_{32}^a] & \text{for } n = 17, \ldots, 32; \\
(p_{13}^a + p_{14}^a) (1 - p_{22}^a) (1 - p_{33}^a) & \text{for } n = 33, \ldots, 48; \\
p_{14}^a (1 - p_{22}^a) (1 - p_{33}^a) & \text{for } n = 49, \ldots, 64. 
\end{cases}$$
As can be seen from above, the steady-state probability for a state differs from one policy to another, complicating the evaluation of the expected value gained from each policy.

The analyses in Sections 3.1 through 3.3 investigate the firm’s preferred action in a deteriorated intermediate state, specifically state 3. In order to develop insight into the actions in an intermediate state, we restrict our analysis to the case where the actions in states 1 and 2 are $P2$ and $P1$, respectively, and the action in state 4 is limited to the standard maintenance action $M1$. This setting enables us to investigate the impact of all four actions available in state 3, i.e., $a_3 \in \{P1, P2, M1, M2\}$. As a result, the analysis in these sections is restricted to choosing between four policies: $A_9 = [P2, P1, P1, M1]$, $A_{11} = [P2, P1, P2, M1]$, $A_{25} = [P2, P1, M1, M1]$, and $A_{27} = [P2, P1, M2, M1]$. We begin the discussion with the firm’s first decision corresponding to whether to produce or maintain the equipment in an intermediate state.

### 3.1 The Choice between Production and Maintenance in an Intermediate State

As the firm manufactures in states 1 and 2, it has to determine whether it should continue to produce when the process deteriorates to state 3, or alternatively, maintain it in the hope that it returns to better states (1 and 2). We develop a critical ratio of the total profit earned from the manufacturing action (profit per unit times the yield) with respect to the maintenance cost in intermediate states. This critical ratio enables the firm to determine which action, production or maintenance, is a better alternative for the state in question. To see this, we compare the following two policies: $A_9 = [P2, P1, P1, M1]$ and $A_{25} = [P2, P1, M1, M1]$. The firm alters its decision only in the third state in these two policies. It is known from (2), (3), and (5) that the steady-state probabilities for states 1, 2 and 4 are different for these two policies despite the fact that they feature the same actions. Similarly, from (4), the steady-state probability for state 3 is also different, and one cannot readily tell whether their relative values increase or decrease. We define $\Delta_{i,j}^{M1,P1}$ as the change in the numerator term of state $i$ when the firm switches from implementing the maintenance action $M1$ to manufacturing product $P1$ in state $j$. The relationship between the numerator terms are expressed as

\[
\hat{\Pi}_1(A_9) = \hat{\Pi}_1(A_{25}) - \Delta_{1,3}^{M1,P1}, \quad \text{where } \Delta_{1,3}^{M1,P1} < \min \left\{ \hat{\Pi}_1(A_{25}), \hat{\Pi}_3(A_{25}) \right\};
\]

\[
\hat{\Pi}_2(A_9) = \hat{\Pi}_2(A_{25}) - \Delta_{2,3}^{M1,P1}, \quad \text{where } \Delta_{2,3}^{M1,P1} < \min \left\{ \hat{\Pi}_2(A_{25}), \hat{\Pi}_3(A_{25}) \right\};
\]

\[
\hat{\Pi}_3(A_9) = \hat{\Pi}_3(A_{25}); \quad \text{and},
\]

\[
\hat{\Pi}_4(A_9) = \hat{\Pi}_4(A_{25}) + \Delta_{4,3}^{M1,P1}, \quad \text{where } \Delta_{4,3}^{M1,P1} < \min \left\{ \hat{\Pi}_3(A_{25}), \hat{\Pi}_4(A_{25}) \right\}.
\]

Using these expressions, the decision maker can develop a critical ratio that determines her preference in state 3.

**Proposition 1** There exists a critical ratio that determines the firm’s choice between manufacturing and maintenance in state 3:

\[
\gamma_{3}^{M1,P1} = -\left( \frac{r_1}{c_3,M1} + \frac{r_2}{c_3,M1} \right) \left( \frac{\hat{\Pi}_1(A_{25}) - \Delta_{1,3}^{M1,P1}}{\Pi_3(A_{25})} \right) - \left( \frac{r_2}{c_3,M1} + \frac{r_3}{c_3,M1} \right) \left( \frac{\hat{\Pi}_2(A_{25}) - \Delta_{2,3}^{M1,P1}}{\Pi_3(A_{25})} \right) + \left( \frac{r_3}{c_3,M1} + \frac{r_4}{c_3,M1} \right) \left( \frac{\hat{\Pi}_4(A_{25}) + \Delta_{3,3}^{M1,P1}}{\Pi_3(A_{25})} \right)
\]

\[
+ \left( \frac{EV(A_{25})}{c_3,M1} \right) \left\{ \frac{\hat{\Pi}_1(A_{25}) - \Delta_{2,3}^{M1,P1}}{\Pi_3(A_{25})} \right\} + \left( \frac{\hat{\Pi}_2(A_{25}) - \Delta_{3,3}^{M1,P1}}{\Pi_3(A_{25})} \right) \left( \frac{\hat{\Pi}_3(A_{25})}{\Pi_3(A_{25})} \right) + \left( \frac{\hat{\Pi}_4(A_{25}) + \Delta_{4,3}^{M1,P1}}{\Pi_3(A_{25})} \right) \right\}.
\]

(6)
If $\gamma_3^{M1,P1} \leq 0$, then $a^*_3 = P1$. However, if $\gamma_3^{M1,P1} > 0$, the optimal decision in state 3 can be determined by comparing $\gamma_3^{M1,P1}$ with $\frac{r_{M1,M1}}{c_{s,M1}}$. a) If $\frac{r_{M1,P1}}{c_{s,M1}} > \gamma_3^{M1,P1}$, then $a^*_3 = P1$ because $EV(A_9 = [P2, P1, P1, M1]) > EV(A_{25} = [P2, P1, M1, M1])$; b) If $\frac{r_{M1,P1}}{c_{s,M1}} < \gamma_3^{M1,P1}$ then $a^*_3 = M1$ because $EV(A_9 = [P2, P1, P1, M1]) < EV(A_{25} = [P2, P1, M1, M1])$; and, c) If $\frac{r_{M1,P1}}{c_{s,M1}} = \gamma_3^{M1,P1}$, then the firm is indifferent between production and maintenance actions in state 3 because $EV(A_9 = [P2, P1, P1, M1]) = EV(A_{25} = [P2, P1, M1, M1])$.

There is an economic interpretation of the critical ratio in (6), which corresponds to the ratio of the profit that can be earned by producing in state 3 relative to the maintenance cost. The value of (6) tells the decision maker the least amount of money that she needs to earn in order to justify manufacturing over maintenance in a deteriorated intermediate state. Thus, the critical ratio $\gamma_3^{M1,P1}$ can be interpreted as the reservation price for the manufacturing option. Because $r_i, c_i \in P > 0$ and $c_{i,M1} > 0$ for all $i$, a critical ratio value that is less than zero implies that the firm benefits more by the manufacturing option than the maintenance alternative. The value of the critical ratio $\gamma_3^{M1,P1}$ increases with: i) lower values of $r_{1,P2}$ and $r_{2,P1}$, i.e., the profit earned from production in states 1 and 2; ii) higher values of $c_{i,M1}$, the cost of the maintenance action in state 4; iii) higher values of $\Delta_{1,3}^{M1,P1}$, $\Delta_{2,3}^{M1,P1}$ and $\Delta_{4,3}^{M1,P1}$, i.e., the change in the numerator terms in states 1, 2, and 4; iv) higher values of expected processing times for production actions: $\tau_{1,P2}$, $\tau_{2,P1}$ and $\tau_{4,M1}$; and v) higher values of $EV(A_{25} = [P2, P1, M1, M1])$, the expected value generated from policy $A_{25}$ featuring the maintenance action in state 3. All five of these conditions imply that the firm needs to earn a higher profit in state 3 in order to justify manufacturing of $P1$ rather than employing the maintenance action $M1$.

### 3.2 The Production Choice in an Intermediate State

This section analyzes the scenario when the decision in the deteriorated state is restricted to manufacturing $P1$ or $P2$. The following two policies can be used in order to develop the critical ratio for the production choice in state 3: $A_9 = [P2, P1, P1, M1]$ and $A_{11} = [P2, P1, P2, M1]$. It was argued earlier that the process is more likely to deteriorate from state 3 to state 4 when product $P2$ is manufactured (rather than product $P1$); thus, $p_{33}^{P2} < p_{33}^{P1}$ and $p_{34}^{P2} > p_{34}^{P1}$. From (2), the firm has $\tilde{\Pi}_1(A_9) = p_{41}^{M1} (1 - p_{22}^{P1}) (1 - p_{33}^{P1}) < \tilde{\Pi}_1(A_{11}) = p_{41}^{M1} (1 - p_{22}^{P1}) (1 - p_{33}^{P2})$ because $(1 - p_{33}^{P1}) < (1 - p_{33}^{P2})$. For the same reason, from (3) and (5), it can be seen that $\tilde{\Pi}_2(A_9) < \tilde{\Pi}_2(A_{11})$ and $\tilde{\Pi}_4(A_9) < \tilde{\Pi}_4(A_{11})$. These observations imply that the numerators of the steady-state probabilities of states 1, 2, and 4 are greater in policy $A_{11}$. As a result, the steady-state probability of state 3 is smaller in policy $A_{11}$. Let us define $\delta_{3}^{P1,P2} = \frac{p_{33}^{P2}}{p_{33}^{P1}}$ as the ratio of the deterioration probabilities from producing the high-end product $P2$ and the low-end product $P1$ for all $1 \leq i < j \leq N$.

From (1), $\delta_{3}^{P1,P2} > 1$ and has a finite value. Then, the relationship between numerator terms can be expressed as follows: $\tilde{\Pi}_1(A_{11}) = \tilde{\Pi}_1(A_9) \times \delta_{3}^{P1,P2}$; $\tilde{\Pi}_2(A_{11}) = \tilde{\Pi}_2(A_9) \times \delta_{3}^{P1,P2}$; $\tilde{\Pi}_3(A_{11}) = \tilde{\Pi}_3(A_9)$; and, $\tilde{\Pi}_4(A_{11}) = \tilde{\Pi}_4(A_9) \times \delta_{3}^{P1,P2}$. The firm can now develop a critical ratio of revenues that determines the production choice.
in the intermediate state.

**Proposition 2** There exists a critical ratio that determines the manufacturing preference in state 3:

\[
\alpha_3^{P1,P2} = \delta_3^{P1,P2} + \text{EV}(A_9 = [P2, P1, P1, M1]) \left( \frac{\tau_3, P2 - \tau_3, P1 \delta_3^{P1,P2}}{\tau_3, P1} \right).
\]

If \(\alpha_3^{P1,P2} \leq 1\), then \(a_3^* = P2\). However, if \(\alpha_3^{P1,P2} > 1\), then the optimal production decision in state 3 can be determined by comparing \(\alpha_3^{P1,P2}\) with \(\frac{\tau_3, P2}{\tau_3, P1}\).\(a\) If \(\frac{\tau_3, P2}{\tau_3, P1} > \alpha_3^{P1,P2}\), then \(a_3^* = P2\) because \(\text{EV}(A_{11} = [P2, P1, P2, M1]) > \text{EV}(A_9 = [P2, P1, P1, M1])\); \(b\) If \(\frac{\tau_3, P2}{\tau_3, P1} < \alpha_3^{P1,P2}\), then \(a_3^* = P1\) because \(\text{EV}(A_{11} = [P2, P1, P2, M1]) < \text{EV}(A_9 = [P2, P1, P1, M1])\); and, \(c\) If \(\frac{\tau_3, P2}{\tau_3, P1} = \alpha_3^{P1,P2}\), then the firm is indifferent between manufacturing \(P1\) and \(P2\) in state 3 because \(\text{EV}(A_{11} = [P2, P1, P2, M1]) = \text{EV}(A_9 = [P2, P1, P1, M1])\).

The critical ratio \(\alpha_3^{P1,P2}\) provides the decision maker with a reservation price corresponding to her manufacturing choices. The firm needs to make at least \(\alpha_3^{P1,P2} \times \tau_3, P1\) in state 3 in order to justify manufacturing the high-end product \(P2\) rather than the standard product \(P1\). A value of \(\alpha_3^{P1,P2}\) that is less than or equal to 1 implies that the firm benefits more by producing \(P2\). The value of the critical ratio increases with: i) higher values of \(\tau_3, P2\), the expected processing time of manufacturing \(P2\) in state 3; ii) lower values of \(\tau_3, P1\), the expected processing time of manufacturing \(P1\) in state 3; and iii) the expected value gained from the policy that features the production of \(P1\). These three observations lead to a higher profit requirement for the firm to switch from manufacturing product \(P1\) to product \(P2\). It can be seen that when \(\frac{\tau_3, P2}{\tau_3, P1} > \delta_3^{P1,P2}\), the firm has to earn more money than \(\delta_3^{P1,P2} \times \tau_3, P1\) by producing \(P2\) in order to switch from policy \(A_9 = [P2, P1, P1, M1]\) to \(A_{11} = [P2, P1, P2, M1]\). Moreover, \(\alpha_3^{P1,P2}\) is less than \(\delta_3^{P1,P2}\) only when \(1 < \frac{\tau_3, P2}{\tau_3, P1} < \delta_3^{P1,P2}\); otherwise the critical ratio is always larger than the change that takes place in the numerators of steady-state probabilities.

It is important to highlight that (7) generalizes the similar critical ratios developed in Kazaz and Sloan (2008). In that paper, the transition probabilities are defined as linearly proportional with the expected processing times. Our transition probabilities, however, are general as no assumption is made regarding their relationship with the expected processing times.

The critical ratio in (7) provides insight into monotone and non-monotone policies. When the firm’s ratio of profits earned in state from producing \(P2\) and \(P1\) is greater than the critical ratio (corresponding to the case when \(\frac{\tau_3, P2}{\tau_3, P1} > \alpha_3^{P1,P2}\)), the firm’s optimal policy is \(A_{11} = [P2, P1, P2, M1]\) with the production action \(P2\) in state 3. Because the firm produces \(P1\) with a lower profit in a better state \((i = 2)\), this case implies that a non-monotone policy is preferred. Because \(\delta_3^{P1,P2}\), \(\text{EV}(A_9 = [P2, P1, P1, M1])\), and \(\tau_3, P1\) are positive, the value of the critical ratio decreases only when \(\tau_3, P2 - \tau_3, P1 \delta_3^{P1,P2} < 0\). If the difference in \(\tau_3, P2\) and \(\tau_3, P1\) is small, the critical ratio decreases with larger values of \(\delta_3^{P1,P2}\), increasing the possibility that the non-monotone policy \(A_{11} = [P2, P1, P2, M1]\) would be preferred. When \(\tau_3, P2 - \tau_3, P1 \delta_3^{P1,P2} > 0\), on the other hand, the firm
has a critical ratio greater than the ratio of deterioration probabilities, i.e., \( \alpha_3^{P1,P2} > \delta_3^{P1,P2} \). In this case, the possibility that the firm would prefer the non-monotone policy \( A_{11} = [P2, P1, P2, M1] \) decreases as \( \delta_3^{P1,P2} \) increases. Thus, the higher the difference in the expected processing times of \( P2 \) and \( P1 \), the more likely that the firm will follow a monotone policy. A detailed discussion on the conditions that lead to monotone and non-monotone policies is provided in Section 4 using a more general problem setting.

3.3 The Maintenance Choice in an Intermediate State

We now present the firm’s maintenance preference in the deteriorated intermediate state. The decision is restricted to performing maintenance actions \( M1 \) and \( M2 \). The following two policies are beneficial in developing the critical ratio for the maintenance choice in state 2: \( A_{25} = [P2, P1, M1, M1] \) and \( A_{27} = [P2, P1, M2, M1] \).

As mentioned earlier, the process is more likely to improve from state 3 to states 1 and 2 when maintenance action \( M2 \) is performed; thus, \( p_{3i}^{M2} < p_{3i}^{M1} \) and \( p_{3i}^{M2} > p_{3i}^{M1} \) for \( i = 1, 2 \). It can be seen from (2)–(5) that the firm has \( \hat{\Pi}_1(A_{25}) < \hat{\Pi}_1(A_{27}) \), \( \hat{\Pi}_2(A_{25}) < \hat{\Pi}_2(A_{27}) \), \( \hat{\Pi}_3(A_{25}) = \hat{\Pi}_3(A_{27}) \) and \( \hat{\Pi}_4(A_{25}) < \hat{\Pi}_4(A_{27}) \). These observations imply that the numerators of the steady-state probabilities of states 1, 2 and 4 are greater in policy \( A_{27} \). Therefore, the steady-state probability of state 3 is smaller in policy \( A_{27} \). Let us define \( \delta_3^{M1,M2} = \frac{p_{32}^{M2}}{p_{31}^{M1}} \) as the ratio of improvement probabilities from utilizing maintenance actions \( M2 \) and \( M1 \) for all \( 1 \leq j < i \leq N \). From (1), \( \delta_3^{M1,M2} > 1 \) and has a finite value. The relationship between the numerator terms can be expressed as follows: \( \hat{\Pi}_1(A_{27}) = \hat{\Pi}_1(A_{25}) \times \delta_3^{M1,M2} \); \( \hat{\Pi}_2(A_{27}) = \hat{\Pi}_2(A_{25}) \times \delta_3^{M1,M2} \); \( \hat{\Pi}_3(A_{27}) = \hat{\Pi}_3(A_{25}) \) and, \( \hat{\Pi}_4(A_{27}) = \hat{\Pi}_4(A_{25}) \times \delta_3^{M1,M2} \). Using these relationships, the firm can develop another critical ratio in order to determine the maintenance choice in the intermediate state.

Proposition 3 There exists a critical ratio that determines the maintenance preference in state 3:

\[
\lambda_3^{M1,M2} = \delta_3^{M1,M2} + EV(A_{25} = [P2, P1, M1, M1]) \left( \frac{\tau_3^{M1,M2}}{c_3,M1} \right).
\]

If \( \lambda_3^{M1,M2} \leq 1 \), then \( a_3^* = M1 \). However, if \( \lambda_3^{M1,M2} > 1 \), then the optimal maintenance decision in state 3 can be determined by comparing \( \lambda_3^{M1,M2} \) with \( \frac{c_{3,M2}}{c_{3,M1}} \). a) If \( \frac{c_{3,M2}}{c_{3,M1}} > \lambda_3^{M1,M2} \), then \( a_3^* = M1 \) because \( EV(A_{27} = [P2, P1, M2, M1]) < EV(A_{25} = [P2, P1, M1, M1]) \); b) If \( \frac{c_{3,M2}}{c_{3,M1}} < \lambda_3^{M1,M2} \), then \( a_3^* = M2 \) because \( EV(A_{27} = [P2, P1, M2, M1]) > EV(A_{25} = [P2, P1, M1, M1]) \); and, c) If \( \frac{c_{3,M2}}{c_{3,M1}} = \lambda_3^{M1,M2} \), then the firm is indifferent between maintenance actions \( M1 \) and \( M2 \) in state 2 because \( EV(A_{27} = [P2, P1, M2, M1]) = EV(A_{25} = [P2, P1, M1, M1]) \).

A similar economic interpretation can be made for the critical ratio \( \lambda_3^{M1,M2} \) representing the ratio of maintenance expenses between a major and a minor maintenance. It provides the decision maker with the maximum amount of money to be spent in order to justify using the major maintenance action \( M2 \) over the standard action \( M1 \). Specifically, \( \lambda_3^{M1,M2} \times c_{3,M1} \) is the highest amount of money that the firm should be
willing to pay for a major maintenance action in a deteriorated intermediate state. If the maintenance cost of $M_2$ is lower than this amount, then the firm prefers to utilize a major maintenance; otherwise, it should continue to use the standard (or minor) maintenance action. The value of the critical ratio $\lambda_3^{M_1,M_2}$ increases with: i) lower values of $\tau_{3,M_2}$, the expected processing time of the major maintenance action $M_2$ in state 3; ii) higher values of $\tau_{3,M_1}$, the expected processing time of the minor maintenance action $M_1$ in state 3; and iii) the expected value gained from the policy that features the minor maintenance action $M_1$ in state 3. These observations lead to a higher upper bound, increasing the likelihood of employing the major maintenance action $M_2$ in the optimal decision. It can be seen that when $\frac{\tau_3}{\tau_3} > \delta_3^{M_1,M_2}$, the firm has to spend less money than $\lambda_3^{M_1,M_2} \times c_{3,M_1}$ for the maintenance action $M_2$ in order to switch from policy $A_{25} = [P_2, P_1, M_1, M_1]$ to $A_{27} = [P_2, P_1, M_2, M_1]$. Moreover, $\lambda_3^{M_1,M_2}$ is less than $\delta_3^{M_1,M_2}$ only when $\frac{\tau_3}{\tau_3} > \delta_3^{M_1,M_2} > 1$; otherwise the critical ratio is always larger than the change that takes place in the numerators of steady-state probabilities.

The critical ratio in (8) provides insight into the maintenance-related monotone and non-monotone policies. When the firm’s ratio of maintenance costs from the major maintenance $M_2$ and the minor maintenance $M_1$ is less than the critical ratio $\lambda_3^{M_1,M_2}$, the firm’s optimal policy is $A_{27} = [P_2, P_1, M_2, M_1]$ with the maintenance action $M_2$ in state 3. Because the firm utilizes $M_1$ with a lower expense in a worse state, this case implies that a non-monotone policy is preferred. Because $\delta_3^{M_1,M_2}$, $EV (A_{25} = [P_2, P_1, M_1, M_1])$, and $c_{3,M_1}$ are positive, the value of the critical ratio increases only when $\tau_3 \cdot \delta_3^{M_1,M_2} - \tau_3 < 0$. In other words, if the difference in $\tau_3$ is small, the critical ratio increases with larger values of $\delta_3^{M_1,M_2}$, making it easier for the firm to prefer the non-monotone policy $A_{27} = [P_2, P_1, M_2, M_1]$. Moreover, when $\tau_3 \cdot \delta_3^{M_1,M_2} - \tau_3 < 0$, the firm has a critical ratio less than the ratio of improvement probabilities, i.e., $\lambda_3^{M_1,M_2} < \delta_3^{M_1,M_2}$. In this case, the possibility that the firm would prefer the non-monotone policy $A_{27} = [P_2, P_1, M_2, M_1]$ becomes increasingly difficult. Thus, the higher the difference in the expected processing times of $M_1$ and $M_2$, the more likely that the firm will follow a monotone policy. A detailed discussion regarding the conditions for monotonicity is provided in Section 4.

### 3.4 Combining the Three Critical Ratios

This section shows how the critical ratios can be combined in order to determine the best policy among the four candidate policies: $A_9 = [P_2, P_1, P_1, M_1]$, $A_{11} = [P_2, P_1, P_2, M_1]$, $A_{25} = [P_2, P_1, M_1, M_1]$, and $A_{27} = [P_2, P_1, M_2, M_1]$. Recall that the critical ratio $\gamma_3^{M_1,P_1}$ enables the firm to choose between the production and maintenance options, $\alpha_3^{P_1,P_2}$ provides the best production alternative, and $\lambda_3^{M_1,M_2}$ reveals the best maintenance action. They help the decision maker to determine the best policy.

**Proposition 4**

a) The best policy is $A_9 = [P_2, P_1, P_1, M_1]$ when $r_3P_1 \geq \left\{ r_3P_1 \frac{1}{\alpha_3^{P_1,M_1}}, c_{3,M_1}^{M_1,P_1}, c_{3,M_2}^{M_1,P_1} \right\}$; 
b) The best policy is $A_{11} = [P_2, P_1, P_2, M_1]$ when $r_3P_2 \geq \left\{ r_3P_2 \frac{1}{\gamma_3^{M_1,P_1}}, c_{3,M_1}^{M_1,P_1}, c_{3,M_2}^{M_1,P_1} \right\}$; 
c) The best policy is $A_{25} = [P_2, P_1, M_1, M_1]$ when $c_{3,M_1} \geq \left\{ r_3P_1 \frac{1}{\gamma_3^{M_1,P_1}}, r_3P_2 \frac{1}{\gamma_3^{M_1,P_1}} \right\}$ and $c_{3,M_1} \leq$
\[ c_{3,M2} \frac{1}{\gamma_3}; \text{d) The best policy is } A_{27} = [P2, P1, M2, M1] \text{ when } c_{3,M2} \geq \left\{ \frac{r_3, P1 \lambda_{M1,M2}}{\tau_3, P3} \lambda_{M1,M2}, \frac{r_3, P2 \lambda_{M1,M2}}{\tau_3, P2} \lambda_{M1,M2} \right\} \text{ and } c_{3,M2} \leq c_{3,M1} \lambda_{M1,M2}. \]

Proposition 4 enables the firm to determine the optimal choice in state 3 when the decisions in other states are restricted to \( a_1 = P2, a_2 = P1 \) and \( a_4 = M1 \). However, the firm has a production choice in state 1 \((a_1 \in P)\), the same four choices in state 2 \((a_2 \in P \cup M)\) and a maintenance choice in state 4 \((a_4 \in M)\). The same set of critical ratios can be developed for other states. It is necessary to develop \( \lambda_{M1,P1}^2 \) for state 2, \( \alpha_1^{P1,P2} \) and \( \alpha_2^{P1,P2} \) for states 1 and 2, and \( \lambda_2^{M1,M2} \) and \( \lambda_4^{M1,M2} \) for states 2 and 4 in order to determine the optimal policy among the previously reported 64 policies. Section 4 presents a comprehensive review of the generalized forms of the critical ratios using an arbitrary number of states.

### 3.5 Incorporating Minimum Production Requirements

An important issue for semiconductor manufacturers is to comply with the market requirements, which corresponds to the firm’s commitment to producing an expected amount of each of its products. The comprehensive list of policies in a four-state problem provided in Table 1 includes many pure product policies such as \( A_1 = [P1, P1, P1, M1] \), where only product P1 is manufactured. When optimal, policy \( A_1 \) implies that product \( P2 \) should not be manufactured. However, it is likely that the firm will be operating under demand constraints, requiring that it manufacture both products. Incorporating a minimum production requirement has a significant impact both on the optimal policy choice and the critical ratios available for comparison.

Under a policy \( A \), the expected production quantities for each product are defined as

\[ Y_{P1} (A) = \sum_{i=1}^{N} y_{i,a} \Pi_i (A) I_{a_i = P1} / \sum_{i=1}^{N} \tau_{i,a} \Pi_i (A) \text{ and } Y_{P2} (A) = \sum_{i=1}^{N} y_{i,a} \Pi_i (A) I_{a_i = P2} / \sum_{i=1}^{N} \tau_{i,a} \Pi_i (A). \]

Based on obligations to downstream electronics manufacturers, for example, the firm might enforce a minimum on the expected production quantity for \( P1 \) and \( P2 \), defined as \( MPR_{P1} \) and \( MPR_{P2} \), respectively, through the following constraints:

\[ Y_{P1} (A) \geq MPR_{P1} \text{ and } Y_{P2} (A) \geq MPR_{P2}, \]  

(9)

The immediate consequence of non-negative \( MPR_{P1} \) and \( MPR_{P2} \) constraints as in (9) is that it reduces the number of potentially optimal policies from 64 to 28 as shown in Table 2. It can be seen that stronger constraints on the minimum production requirements reduces this number even further.

The firm can increase its throughput in three different ways. First, it can switch from maintenance to production in a deteriorated state. Consider the event that \( r_{3,P1}/c_{3,M1} < \gamma_3^{M1,P1} \), so the optimal action in state 3 from an economic perspective is \( M1 \) (by Proposition 3). If the expected production in (9) is not satisfied for \( P1 \), the firm might choose to manufacture \( P1 \) instead of performing maintenance in state 3. The comparison of the expected production from the policies used in Proposition 1, \( A_{25} = [P2, P1, M1, M1] \) and \( A_9 = [P1, P1, P1, M1] \), provides the conditions for increasing the throughput. Second, switching to a major maintenance from a minor maintenance action can increase throughput in better states. The comparison of
comparison of the expected production from policies frequency of manufacturing. This requires swapping of production and maintenance in a better state. The production. Third, the firm can perform maintenance in an earlier (better) state in order to increase the

cases can be written for any base policy.

<table>
<thead>
<tr>
<th>Group 1:</th>
<th>Group 2:</th>
<th>Group 3:</th>
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<tr>
<td>A₃ = [P₁, P₁, P₂, M₁]</td>
<td>A₂₁ = [P₁, P₂, M₁, M₁]</td>
<td>A₃₅ = [P₁, M₁, P₂, M₁]</td>
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<tr>
<td>A₄ = [P₁, P₁, P₂, M₂]</td>
<td>A₂₂ = [P₁, P₂, M₁, M₂]</td>
<td>A₃₆ = [P₁, M₁, P₂, M₂]</td>
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<tr>
<td>A₅ = [P₁, P₂, P₁, M₁]</td>
<td>A₂₃ = [P₁, P₂, M₂, M₁]</td>
<td>A₃₉ = [P₁, M₂, P₂, M₁]</td>
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<td>A₆ = [P₁, P₂, P₁, M₂]</td>
<td>A₂₄ = [P₁, P₂, M₂, M₂]</td>
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<tr>
<td>A₇ = [P₁, P₂, P₂, M₁]</td>
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<td>A₈ = [P₁, P₂, P₂, M₂]</td>
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policies A₂₅ = [P₂, P₁, M₁, M₁] and A₂₇ = [P₂, P₁, M₂, M₁] shows the conditions to increase the expected production. Third, the firm can perform maintenance in an earlier (better) state in order to increase the frequency of manufacturing. This requires swapping of production and maintenance in a better state. The comparison of the expected production from policies A₂₅ = [P₂, P₁, M₁, M₁] and A₄₁ = [P₂, M₁, P₁, M₁] provides this effect. Note that there is a double switch, from P₁ to M₁ in state 2 and from M₁ to P₁ in state 3, in this comparison. The following proposition shows what conditions lead to increased throughput. For consistency, all of the conditions are stated in terms of policy A₂₅ = [P₂, P₁, M₁, M₁]; however, similar conditions can be written for any base policy.

**Proposition 5** The firm can increase its expected production of P₁ by: a) switching from maintenance to production in an intermediate state, e.g., state 3, when \( y_{3, P₁} - y_{2, P₁} \left( \Delta_{M₁, P₁}^{M₁, P₁} / \hat{Π}_{3} (A₂₅) \right) > \)

\[
Y_{P₁} (A₂₅) \left\{ -τ₁₁, P₂ \left( Δ_{1,3}^{M₁, P₁} / \hat{Π}_{3} (A₂₅) \right) - τ₁₂, P₁ \left( Δ_{2,3}^{M₁, P₁} / \hat{Π}_{3} (A₂₅) \right) + (τ₂₁, P₁ - τ₂₃, M₁) + τ₄₁, M₁ \right\}; \]

b) applying major maintenance, rather than minor maintenance, in an intermediate state when \( τ₃ₐ₁, M₂ - τ₃ₐ₂, M₁ δ₃₁₄ M₂ < 0 \). And, c) the firm can increase the throughput of a product, e.g., P₂, by performing maintenance in a better state when \( y_{1, P₂} > \)

\[
\frac{Y_{P₂}(A₂₅)}{(Δ_{1,2}^{M₁, P₁} - Δ_{1,3}^{M₁, P₁})} \left\{ τ₁₁, P₂ \left( Δ_{1,2}^{M₁, P₁} - Δ_{1,3}^{M₁, P₁} \right) + (τ₂₁, M₁ - τ₂₃, P₁) \left( Δ_{2,3}^{M₁, P₁} / \hat{Π}_{2} (A₂₅) \right) + (τ₃₁, P₁ - τ₃ₐ₂, M₁) \left( Δ_{2,3}^{M₁, P₁} / \hat{Π}_{1} (A₂₅) \right) + τ₄₂, M₁ \left( Δ_{1,2}^{M₁, P₁} - Δ_{1,3}^{M₁, P₁} \right) \right\}.
\]

Three conclusions can be made from the above proposition. Part a) shows that the firm can increase its expected production of P₁ when the yield in state 3 is relatively close to that of state 2 and the changes in steady-state probabilities do not increase the adjusted expected total processing time (right hand side of the condition). This condition is not satisfied when the firm is spending too much time performing maintenance, resulting in lower throughput. Part b) proves that the firm can increase the throughput of a product by switching from minor maintenance to major maintenance. This occurs when the expected processing time of M2 is smaller than that of M1 multiplied by the increase in improvement probabilities. Part c) shows that
the firm can also increase the throughput of a product by performing its maintenance in an earlier state. This requires that adjustment in the total expected processing time is not significant. Considering the results in parts b) and c), it can be concluded that the frequency and timing of maintenance can play a strategic role in increasing the throughput of a product. This can be accomplished by either performing major maintenance or performing maintenance before the process becomes highly deteriorated.

A comparison of the expected production quantities reveal that policies in Group 3 result in the lowest expected production levels for each product. Specifically, $A_{45} = [P2, M2, P1, M1]$ and $A_{39} = [P1, M2, P2, M1]$ provide the smallest $Y_{P1}(A)$ and $Y_{P2}(A)$, respectively. Consider the event that $Y_{P1}(A_{45})$ does not satisfy (9), making $A_{45}$ infeasible. Then, the next three policies with the smallest expected production quantities are $A_{46} = [P2, M2, P1, M2]$, $A_{41} = [P2, M1, P1, M1]$, and $A_{42} = [P2, M1, P1, M2]$. If these three policies also fail to satisfy (9), then the decision maker has to consider producing $P1$ at the latest in state 2 or better (state 1), eliminating these four policies from further consideration. The following proposition shows that the comparison of the maximum $Y_{P1}(A)$ and $Y_{P2}(A)$ from policies that feature manufacturing of each product in a single state with $MPRP_1$ and $MPRP_2$ leads to an effective set of structural properties.

**Proposition 6** Under the conditions established in Proposition 5 for increasing throughput a) If $MPRP_1 > Y_{P1}(A_{42})$ and $MPRP_2 > Y_{P2}(A_{36})$, then no policy in Group 3 can be optimal. Thus, the number of potentially optimal policies reduces to 20 and the critical ratios $\gamma_2^{M1,P1}$ and $\lambda_2^{M1,M2}$ can be eliminated from the problem. b) If $MPRP_1 > Y_{P1}(A_{24})$ or $MPRP_2 > Y_{P2}(A_{28})$, then no policy in Group 2 can be optimal. Thus, the number of potentially optimal number of policies reduces to 20 and the critical ratios $\gamma_3^{M1,P1}$ and $\lambda_3^{M1,M2}$ can be eliminated from the problem. c) If $MPRP_1 > Y_{P1}(A_{40})$ or $MPRP_2 > Y_{P2}(A_{46})$, then no policy in Groups 2 and 3 can be optimal. These conditions reduce the potentially optimal number of policies to 12 and the critical ratios $\gamma_2^{M1,P1}$, $\lambda_2^{M1,M2}$, $\gamma_3^{M1,P1}$ and $\lambda_3^{M1,M2}$ can be eliminated from the problem. Moreover, d) when $MPRP_1 > Y_{P1}(A_{40})$ policies $A_3$ through $A_8$ in Group 1 and when $MPRP_2 > Y_{P2}(A_{46})$ policies $A_9$ through $A_{14}$ in Group 1 cannot be optimal, eliminating the critical ratios $\alpha_1^{P1,P2}$, $\gamma_2^{M1,P1}$, $\lambda_2^{M1,M2}$, $\gamma_3^{M1,P1}$ and $\lambda_3^{M1,M2}$ from the problem.

The above proposition provides insight into the influence of the firm’s minimum production requirements. First, it shows that stronger constraints reduce the number of potentially optimal policies, resulting in a smaller search for the optimal policy. Second, it highlights the relationship between the minimum production requirements and the three sets of critical ratios established in Sections 3.1 through 3.3. Stronger production requirements in (9) result in a smaller set of necessary critical ratios in determining the optimal policy. Example 1 in the Appendix demonstrates the impact of the minimum production requirements on the optimal policy choice.

It should be highlighted here that 12 of the above 28 policies are production-related monotone policies, i.e.
as the state gets worse, the firm does not switch to a more profitable product: $A_3$, $A_4$, $A_7$, $A_8$, $A_{21}$, $A_{22}$, $A_{23}$, $A_{24}$, $A_{51}$, $A_{52}$, $A_{55}$ and $A_{56}$. However, four of these twelve policies violate this behavior from a maintenance perspective because the firm switches to major maintenance as the state improves: $A_{23}$, $A_{24}$, $A_{55}$ and $A_{56}$. As a result, there are only eight pure monotone policies that comply with production and maintenance action switches. Section 4 develops the general conditions for the optimality of production- and maintenance-related monotone and non-monotone policies.

Our goal in this paper is to present the structural properties of the problem and provide insight into the firm’s decisions with the use of critical ratios. Therefore, we next focus on how these critical ratios change in problem settings with an arbitrary number of states.

4 Generalizing the Critical Ratios

The three sets of critical ratios developed in the previous section can be generalized to a problem setting that features $N$ states. Let us begin our discussion with the critical ratio that determines the firm’s preference between the manufacturing and maintenance alternatives.

4.1 Critical Ratios for Switching between Production and Maintenance

We consider the policy $A_n = [a_1, \ldots, a_N]$ with action $a_j = M1$ in state $j$ and the firm needs to determine whether to switch its action to $a_j = P1$. To establish the general form of the critical ratio, it is necessary to highlight the following three changes in the numerator terms: 1) the numerator for the steady-state probability of state $j$ remains the same, 2) the numerator term in states $i = 1, \ldots, j-1$ decreases with $\hat{\Pi}_i(A_n) - \Delta_{i,j}^{M1,P1}(A_n)$, and 3) the numerator term in states $i = j+1, \ldots, N$ increases with $\hat{\Pi}_i(A_n) + \Delta_{i,j}^{M1,P1}(A_n)$. As a result of these observations, the critical ratio corresponding to the choice between production and maintenance can be expressed as follows:

$$
\gamma_{ij}^{M1,P1} = \sum_{i=1}^{j-1} \left[ -1_{a_i \in P} \left( \frac{r_{i,a_i}}{c_{j,M1}} \right) + 1_{a_i \in M} \left( \frac{c_{i,a_i}}{c_{j,M1}} \right) \left( \frac{\hat{\Pi}_i(A_n) - \Delta_{i,j}^{M1,P1}(A_n)}{\hat{\Pi}_j(A_n)} \right) \right]
+ \sum_{i=j+1}^{N} \left[ -1_{a_i \in P} \left( \frac{r_{i,a_i}}{c_{j,M1}} \right) + 1_{a_i \in M} \left( \frac{c_{i,a_i}}{c_{j,M1}} \right) \left( \frac{\hat{\Pi}_i(A_n) + \Delta_{i,j}^{M1,P1}(A_n)}{\hat{\Pi}_j(A_n)} \right) \right]
+ \left( \frac{EV(A_n)}{c_{j,M1}} \right) \left\{ \sum_{i=1}^{j-1} \tau_{i,a_i} \left( \frac{\hat{\Pi}_i(A_n) - \Delta_{i,j}^{M1,P1}(A_n)}{\hat{\Pi}_j(A_n)} \right) + \tau_{j,P1} \sum_{i=j+1}^{N} \tau_{i,a_i} \left( \frac{\hat{\Pi}_i(A_n) + \Delta_{i,j}^{M1,P1}(A_n)}{\hat{\Pi}_j(A_n)} \right) \right\}. \quad (10)
$$

It should be emphasized that policy $A_n$ does not have to be a monotone policy, and (10) captures the policy improvement behavior regardless of the type of the policy. The critical ratios can be used to derive conditions under which a control-limit policy is optimal. Reflecting the operating environment of a semiconductor manufacturer, the firm is expected to earn less profit as the process deteriorates, and spend more money on mainte-
nance in deteriorated states. Therefore, let us consider the case that \( \frac{r_{j,P1}}{c_{j,M1}} \) is decreasing in \( j \). The relationship between \( \frac{r_{j,P1}}{c_{j,M1}} \) with \( \gamma_{j}^{M1,P1} \) in a state \( j \) establishes a set of sufficient conditions for an optimal control-limit policy. Specifically, when the decrease in \( \gamma_{j}^{M1,P1} \) as the process condition deteriorates is greater than the decrease in \( \frac{r_{j,P1}}{c_{j,M1}} \), the firm is guaranteed to have a monotone optimal policy. This is because the relative values of the critical ratio \( \gamma_{j}^{M1,P1} \) and \( \frac{r_{j,P1}}{c_{j,M1}} \) can switch their sign only once. Suppose product \( P1 \) is manufactured in states 1 through \( j-1 \), and maintenance action \( M1 \) is performed in states \( j+2 \) through \( N \). Let us define the following two policies: \( A_{j} = [a_{1},...,a_{j-1} = P1, a_{j},...,a_{N} = M1] \) and \( A_{j+1} = [a_{1},...,a_{j} = P1, a_{j+1},...,a_{N} = M1] \). When the firm considers the switch from maintenance to production in state \( j \), the base policy is \( A_{j} \), and when it considers the switch in state \( j+1 \), the base policy is \( A_{j+1} \). The following proposition establishes sufficient conditions for a monotone optimal policy.

**Proposition 7** If the following conditions are satisfied, then there exists a threshold state, \( j \), such that production is the optimal choice for all states \( j < j \), and maintenance is optimal for all states \( j \geq j \).

\[
\sum_{i=1}^{j} [\tau_{i,P1} \Delta_{i,j+1}^{M1,P1} (A_{j+1})] \leq \left[ \sum_{i=1}^{j} [\tau_{i,P1} \Delta_{i,j+1}^{M1,P1} (A_{j+1})] + \sum_{i=j+2}^{N} [c_{i,M1} \Delta_{i,j+1}^{M1,P1} (A_{j+1})] \right] \tag{11}
\]

\[
r_{j+1,P1} + c_{j+1,M1} \leq \left[ \sum_{i=1}^{j} [\tau_{i,P1} \bar{\pi}_{i} (A_{j+1})] + \sum_{i=j+2}^{N} [\tau_{i,M1} \bar{\pi}_{i} (A_{j+1})] \right] \left( EV (A_{j+1}) - EV (A_{j}) \right) \tag{12}
\]

\[
\frac{r_{j,P1}}{c_{j,M1}} \text{ is decreasing in } j. \tag{13}
\]

The above proposition proves that if it is optimal to maintain in state \( j \), then it is optimal to maintain in states \( j+1 \) through \( N \); this fact can greatly reduce the number of potentially optimal policies. In Section 3.1, it has been concluded that the maintenance action is more desirable in deteriorated intermediate states when 1) the value of the change \( \Delta_{i,j+1}^{M1,P1} \) in the steady-state expressions is large, 2) the profits and the maintenance costs increase, and 3) when the change in the expected processing times is large. These observations are captured in the sufficient conditions in (11) and (12). Condition (11) states that the total change in the sum of the expected processing times due to the switch from maintenance to production should not be larger than the total change that occurs in profits and maintenance costs. Condition (12) focuses on the sum of the profit and the maintenance cost in the state in question, and requires it to be less than or equal to the difference in expected values of the two monotone policies adjusted with normalized expected processing times. Thus, with (11) and (12), any drastic change in profits, maintenance costs, and the expected processing times are prevented, ensuring that the firm does not switch to maintenance and back to production again.
Sufficient conditions in (11)–(13) generalize those reported in the literature. Sloan (2008) provides five sufficient conditions that are required collectively. The conditions can be summarized as: C1) the profits $r_{j,P_1}$ are decreasing in $j$, and the costs $c_{j,M_1}$ are increasing in $j$; C2) the machine state has increasing failure rate, i.e., $\sum_{j=l}^{N} p_{ij}^{M_1}$ is increasing in $i$ for $l = 1,2,\ldots,N$ and $a \in \{P_1,M_1\}$; C3) $c_{j,M_1} - r_{j,P_1}$ is increasing in $j$; C4) for each state $l$, the sum of the state transition probability matrices is subadditive, i.e., $\sum_{j=1}^{N} p_{ij}^{M_1} - \sum_{j=1}^{N} p_{ij}^{P_1}$ is decreasing in $i$ for all $l = 1,\ldots,N$; and C5) the expected completion times are subadditive, i.e., $\tau_{j,M_1} - \tau_{j,P_1}$ is decreasing in $j$. Note that (13) is not as restrictive as condition C1 — the profits and the maintenance costs may increase or decrease with respect to $j$. Conditions (11) and (12) are significantly less restrictive than the subadditivity requirements in their paper, described by conditions C3 and C4. Therefore, the sufficient conditions provided for monotonicity in this paper generalize those reported in the literature.

### 4.2 Production-related Critical Ratios

For the production choices, let us consider the policy $A_n = [a_1,\ldots,a_N]$ with the action $a_j = P_1$ in state $j$. It can be observed that when the firm switches its action from $P_1$ to $P_2$ in state $j$, the numerator term for state $j$ remains the same, and the numerator terms for all the other states increase; thus, $\delta_{i}^{P_1,P_2} > 0$ for all $i = 1,\ldots,N$ where $i \neq j$. As a result of this observation, the critical ratio for production choices in a $N$-state problem is as follows:

$$\alpha_{j}^{P_1,P_2} = \delta_{j}^{P_1,P_2} + \text{EV}(A_n) \left( \frac{\tau_{j,P_2} - \tau_{j,P_1} \delta_{j}^{P_1,P_2}}{r_{j,P_1}} \right).$$

(14)

A non-monotone policy among production choices can be observed when $r_{i,P_2} / r_{i,P_1} < \alpha_{j}^{P_1,P_2}$ in state $i$ and $r_{i,P_2} / r_{j,P_1} > \alpha_{j}^{P_1,P_2}$ in state $j$ where $1 \leq i < j \leq N$. This implies that the firm prefers to manufacture the low-end product $P_1$ with lower profit in a better state $i$ and the high-end product $P_2$ with a higher profit in a deteriorated state $j$. Even if the ratio of profits in each state is constant, an increasing behavior of $\alpha_{j}^{P_1,P_2}$ in $j$ can create this scenario. Therefore, it is beneficial to establish the conditions for the increasing and decreasing behavior of the production-related critical ratio in (14).

**Proposition 8**  
a) $\alpha_{j}^{P_1,P_2}$ is increasing in $j$ when the following three conditions are satisfied: 1) $\delta_{j}^{P_1,P_2}$ is increasing in $j$, 2) $\frac{\tau_{j,P_2}}{r_{j,P_1}} > \delta_{j}^{P_1,P_2}$ for each $j = 1,\ldots,N$, and 3) $\frac{\tau_{j,P_2} - \tau_{j,P_1} \delta_{j}^{P_1,P_2}}{r_{j,P_1}}$ is increasing in $j$;  
b) $\alpha_{j}^{P_1,P_2}$ is decreasing in $j$ when the following three conditions are satisfied: 1) $\delta_{j}^{P_1,P_2}$ is decreasing in $j$, 2) $\frac{\tau_{j,P_2}}{r_{j,P_1}} < \delta_{j}^{P_1,P_2}$ for each $j = 1,\ldots,N$, and 3) $\frac{\tau_{j,P_2} - \tau_{j,P_1} \delta_{j}^{P_1,P_2}}{r_{j,P_1}}$ is decreasing in $j$.

The increasing behavior of $\alpha_{j}^{P_1,P_2}$ through the above three conditions is useful in establishing a set of sufficient conditions for a monotone optimal policy with respect to production choices. It should be observed that when $r_{j,P_1}$ is decreasing in $j$, $\alpha_{j}^{P_1,P_2}$ is increasing in $j$ under conditions 1 and 2, and condition 3 is not necessary as it is automatically satisfied. Considering the operating environment for a semiconductor manufacturer, it can be expected to have the profits decrease as the process deteriorates, and therefore the
firm’s \( \alpha_{j}^{P_1,P_2} \) is increasing in \( j \) under less restrictive conditions (1 and 2). In the event that \( r_{j,P_1} \) is increasing in \( j \), \( \alpha_{j}^{P_1,P_2} \) is still increasing in \( j \) under conditions 1, 2, and 3 together. It is obvious that a monotone optimal policy is ensured when \( \frac{r_{j,P_2}}{r_{j,P_1}} < \alpha_{j}^{P_1,P_2} \) for all \( j = 1, \ldots, N-1 \), or when \( \frac{r_{j,P_2}}{r_{j,P_1}} > \alpha_{j}^{P_1,P_2} \) for all \( j = 1, \ldots, N-1 \). Monotonicity is warranted when the ratio of profits is greater than the critical ratio in better states and crosses under the critical ratio only once.

**Proposition 9** The following set of sufficient conditions leads to a monotone policy with respect to production choices: 1) \( \frac{r_{j,P_2}}{r_{j,P_1}} \) is decreasing, 2) \( \delta_{j}^{P_1,P_2} \) is increasing in \( j \), 3) \( \frac{r_{j,P_2}}{r_{j,P_1}} > \delta_{j}^{P_1,P_2} \) for each \( j = 1, \ldots, N-1 \), and 4) \( \frac{r_{j,P_2} - r_{j,P_1} \delta_{j}^{P_1,P_2}}{r_{j,P_1}} \) is increasing in \( j \).

The above sufficient conditions generalize those reported in the literature significantly. Sloan (2008) reports five sufficient conditions, related to those discussed above (immediately following Proposition 7). In addition to conditions C1 and C2, the following three conditions are required: C3’ \( r_{j,a_j}/[1 - \theta_{jj}^{a_j}] \) is superadditive for \( a_j \in \mathbf{P} \), i.e., \( r_{j,P_1}[1 - \theta_{jj}^{P_1}] - r_{j,P_2}[1 - \theta_{jj}^{P_2}] \) is increasing in \( j \); C4’ for each state \( l \), the sum of the state transition probability matrices is subadditive, i.e., \( \sum_{j=1}^{N} p_{1,j}^{P_1} / [1 - \theta_{jj}^{P_1}] - \sum_{j=1}^{N} p_{2,j}^{P_2} / [1 - \theta_{jj}^{P_2}] \) is decreasing in \( i \) for all \( l = 1, \ldots, N \); and C5’ the expected processing times are subadditive, i.e., \( r_{j,P_1}/[1 - \theta_{jj}^{P_1}] - r_{j,P_2}/[1 - \theta_{jj}^{P_2}] \) is decreasing in \( j \). Proposition 9 does not require anything like condition C1 — the profits may increase or decrease with respect to \( j \). This can be seen in the case when \( r_{j,P_1} \) and \( r_{j,P_2} \) are increasing in \( j \) with the ratio of profits \( \frac{r_{j,P_2}}{r_{j,P_1}} \) being constant between states; this violates C1. Under the conditions where \( \alpha_{j}^{P_1,P_2} \) is also constant in each state with a value greater than \( \frac{r_{j,P_2}}{r_{j,P_1}} \), however, our sufficient conditions detect the monotone policy. Moreover, our first condition is less restrictive than condition C3’. Condition C4’ is also more limiting than our third condition. Our second and fourth conditions together are still more general than the subadditivity requirements in their paper. Therefore, the sufficient conditions provided for monotonicity in this paper generalize those reported in the literature. Example 2 provided in the Appendix illustrates a problem for which the sufficient conditions of Sloan (2008) are not met but for which the optimal policy is monotone with respect to the production choices.

### 4.3 Maintenance-related Critical Ratios

A similar critical ratio for the maintenance decision can be determined by considering the policy \( \mathbf{A}_n = [a_1, \ldots, a_N] \) with the action \( a_j = M1 \) in state \( j \). It can be observed that when the firm switches its action from \( M1 \) to \( M2 \) in state \( j \), the numerator term for state \( j \) remains the same, and the numerator terms for all the other states increase, i.e., \( \delta_{i}^{M1,M2} > 0 \) for all \( i = 1, \ldots, N \) where \( i \neq j \). Therefore, the maintenance critical ratio for the \( N \)-state problem can be expressed as follows:

\[
\lambda_j^{M1,M2} = \delta_j^{M1,M2} + EV(\mathbf{A}_n) \left( \frac{\tau_{j,M1} \delta_j^{M1,M2} - \tau_{j,M2}}{c_{j,M1}} \right),
\]

(15)
Among maintenance choices, a non-monotone policy can be observed when \( \frac{c_{i,M2}}{c_{i,M1}} > \lambda_j^{M1,M2} \) in state \( i \) and \( \frac{c_{i,M2}}{c_{i,M1}} > \lambda_j^{M1,M2} \) in state \( j \) where \( 1 \leq i < j \leq N \). This implies that the firm prefers to perform the major maintenance action \( M2 \) with a higher expense in a better state \( i \) and the minor maintenance action \( M1 \) with a lower cost in a deteriorated state \( j \). Even if the ratio of maintenance costs in each state is constant, an increasing behavior of \( \lambda_j^{M1,M2} \) in \( j \) can create this scenario. Therefore, it is beneficial to establish the conditions for the increasing/decreasing behavior of the maintenance-related critical ratio.

**Proposition 10**  

a) \( \lambda_j^{M1,M2} \) is increasing in \( j \) when the following three conditions are satisfied: 1) \( \delta_j^{M1,M2} \) is increasing in \( j \), 2) \( \frac{\tau_{j,M2}}{\tau_{j,M1}} < \delta_j^{M1,M2} \) for each \( j = 1, \ldots, N \), and 3) \( \frac{\tau_{j,M1} \delta_j^{M1,M2} - \tau_{j,M2}}{c_{j,M1}} \) is increasing in \( j \); 
b) \( \lambda_j^{M1,M2} \) is decreasing in \( j \) when the following three conditions are satisfied: 1) \( \delta_j^{M1,M2} \) is decreasing in \( j \), 2) \( \frac{\tau_{j,M2}}{\tau_{j,M1}} > \delta_j^{M1,M2} \) for each \( j = 1, \ldots, N \), and 3) \( \frac{\tau_{j,M1} \delta_j^{M1,M2} - \tau_{j,M2}}{c_{j,M1}} \) is decreasing in \( j \).

The decreasing behavior of \( \lambda_j^{M1,M2} \) through the above three conditions is useful in establishing a set of sufficient conditions for a monotone policy with respect to maintenance choices. It should be observed that when \( c_{j,M1} \) is increasing in \( j \), \( \lambda_j^{M1,M2} \) is decreasing in \( j \) under conditions 1 and 2 (of part b), and condition 3 is not necessary as it is automatically satisfied. Considering the operating environment for a semiconductor manufacturer, maintenance costs can be expected to increase as the process deteriorates, and therefore the firm’s \( \lambda_j^{M1,M2} \) is decreasing in \( j \) under less restrictive conditions (1 and 2). In the event that \( c_{j,M1} \) is decreasing in \( j \), \( \lambda_j^{M1,M2} \) is still decreasing in \( j \) under conditions 1, 2, and 3 together. It is easy to observe that a monotone policy is ensured when \( \frac{c_{i,M2}}{c_{i,M1}} < \lambda_j^{M1,M2} \) (or when \( \frac{c_{i,M2}}{c_{i,M1}} > \lambda_j^{M1,M2} \) for all \( j = 2, \ldots, N \)). Once again, monotonicity is warranted when \( \frac{c_{i,M2}}{c_{i,M1}} \) is greater than \( \lambda_j^{M1,M2} \) in better states and crosses under the critical ratio only once.

**Proposition 11**  
The following set of sufficient conditions leads to a monotone policy with respect to maintenance choices: 1) \( \frac{c_{i,M2}}{c_{i,M1}} \) is decreasing in \( j \), 2) \( \delta_j^{M1,M2} \) is increasing in \( j \), 3) \( \frac{\tau_{j,M2}}{\tau_{j,M1}} < \delta_j^{M1,M2} \) for each \( j = 1, \ldots, N \), and 4) \( \frac{\tau_{j,M1} \delta_j^{M1,M2} - \tau_{j,M2}}{c_{j,M1}} \) is increasing in \( j \).

The above sufficient conditions generalize those reported in the literature significantly. Sloan (2008) extends the maintenance policy results of Hopp and Wu (1990), and requires collectively: C1) the costs \( c_{j,M1} \) and \( c_{j,M2} \) are non-decreasing in \( j \), C2) the machine state has increasing failure rate, C3) \( c_{i,a,c} \in \mathbb{M} \) is superadditive, i.e., the difference in the costs \( c_{j,M1} - c_{j,M2} \) is increasing in \( j \); C4) for state \( l \), the sum of the state transition probability matrices is subadditive, i.e., \( \sum_{j=1}^{N} \left( p_{ij}^{M2} - p_{ij}^{M1} \right) \) is decreasing in \( i \) for all \( l = 1, \ldots, N \), and C5) the expected maintenance times are subadditive, i.e., \( \tau_{j,M1} - \tau_{j,M2} \) is decreasing in \( j \). The conditions listed in Proposition 11 are much more general. Condition C3 is similar to our first condition; however, ours is less restrictive. Similarly, condition C4 is similar — but more restrictive — than our second and fourth conditions combined. In the Appendix, Example 3 illustrates the situation in which some of the Sloan (2008) conditions are not met but for which the optimal policy is monotone with respect to the maintenance actions.
Let us define \( \hat{\alpha}_{P_1, M_1} \) as the minor maintenance threshold with respect to the standard product \( P_1 \) and \( \hat{\alpha}_{P_1, M_2} \) as the major maintenance threshold with respect to \( P_1 \). Proposition 11 is equivalent to saying that \( \hat{\alpha}_{P_1, M_1} \leq \hat{\alpha}_{P_1, M_2} \); this fact can reduce the set of potentially optimal policies drastically. Specifically, the decision maker does not have to consider all four actions in each of the \( N - 2 \) states. In states \( \hat{\alpha}_{P_1, M_2} \) through \( N \), for example, the choice is restricted to be between \( M_1 \) and \( M_2 \), and in states between \( \hat{\alpha}_{P_1, M_1} \) and \( \hat{\alpha}_{P_1, M_2} \) the choice is restricted to \( P_1, P_2 \) or \( M_1 \).

5 Conclusions

This paper considers a manufacturer’s production and maintenance choices under deteriorating process conditions. The firm has to make three decisions in each machine state: 1) whether to produce or maintain the process, 2) if production is chosen, which product to manufacture, and 3) if maintenance is elected, whether to employ a major or a minor maintenance action. Each of these three decisions has trade-offs. In the first, production speeds the process deterioration, and maintenance is likely to improve it; however, while production earns profits, maintenance leads to a cost. The choice of the product also influences the process deterioration: a high-end product provides a higher profit, but takes longer to manufacture and accelerates the process deterioration. A low-end product brings less profit, but has a lower probability of process deterioration. The maintenance choice influences the process improvement similarly. A major maintenance action incurs a higher cost but is more likely to improve the process than a minor maintenance which costs less money. Excessive maintenance, however, can actually reduce net throughput by devoting more time to maintenance rather than production. The paper develops a model that captures the complex relationships between these three decisions, process deterioration and improvement probabilities, profits and costs, and the expected processing times. We incorporate market demand considerations by requiring the firm to satisfy a minimum production level for each product and demonstrate how this influences the optimal policy.

The paper makes four sets of contributions. First, it develops three critical ratios. The first critical ratio determines whether the firm should manufacture or maintain the equipment. The second critical ratio enables the firm to choose the preferred product in each state. The third critical ratio informs the decision maker about the appropriate maintenance action. These critical ratios have economic interpretations. The first two critical ratios can be interpreted as the reservation price, i.e., the maximum amount of money the decision maker should be willing to pay in order to switch from maintenance to production in the first, and from a low-end product to a high-end product in the second. The third critical ratio enables the decision maker to establish an upper bound on the cost of the major maintenance action corresponding to the maximum amount of money she should be willing to spend. Second, the paper shows how these three critical ratios can be combined in order to determine the optimal action among all possible choices. The combination enables the firm to capture the above trade-offs simultaneously. These critical ratios are then generalized to problem

22
settings that feature an arbitrary number of machine states. Third, the paper demonstrates the influence of minimum production requirements on the choice of the optimal policy and the critical ratios used to determine the optimal solution. It shows that, depending on the length of its expected time, maintenance can play a strategic role in increasing the throughput of a high demand product. Fourth, a set of sufficient conditions are developed that lead to monotone optimal policies. These conditions are demonstrated to be significantly more generalized than those reported in the literature. Monotone policies suggest that the firm manufacture its high-technology products in better process conditions and switch to low-technology products as the machine deteriorates. At some level of deterioration, production is no longer viable, and maintenance is performed. The firm should employ a minor maintenance with continued deterioration, and perform a major maintenance in significantly deteriorated states.

References


Online Appendix

for

The Impact of Process Deterioration on Production and Maintenance Policies

September 20, 2010

Proof of Proposition 1: We first derive the critical ratio expression by equating the expected values of the two policies $A_9 = [P_2, P_1, P_1, M1]$ and $A_{25} = [P_2, P_1, M1, M1]$. Recall that

$$
\tilde{\Pi}_1 (A_9) = \tilde{\Pi}_1 (A_{25}) - \Delta^{M1,P1}_{1,3}
$$

(16)

$$
\tilde{\Pi}_2 (A_9) = \tilde{\Pi}_2 (A_{25}) - \Delta^{M1,P1}_{2,3}
$$

(17)

where $0 < \Delta^{M1,P1}_{1,3} < \tilde{\Pi}_1 (A_{25})$ and $0 < \Delta^{M1,P1}_{2,3} < \tilde{\Pi}_2 (A_{25})$ are the changes in the numerator terms of states 1 and 2 when the firm switches from implementing the maintenance action $M1$ to manufacturing product $P1$ in state 3.

The steady-state probabilities of state 3 are not equal for these two policies, the numerator expressions are equal, i.e., $\tilde{\Pi}_3 (A_9) = \tilde{\Pi}_3 (A_{25})$. The worst state resembles the best state with some changes:

$$
\tilde{\Pi}_4 (A_9) = \tilde{\Pi}_4 (A_{25}) + \Delta^{M1,P1}_{4,3}
$$

(18)

where $\Delta^{M1,P1}_{4,3} > 0$.

$$
\begin{align*}
EV (A_9 = [P_2, P_1, P_1, M1]) & = EV (A_{25} = [P_2, P_1, M1, M1]) \\
& = \begin{bmatrix}
\tau_1, p_2 \tilde{\Pi}_1 (A_9) + r_2, p_1 \tilde{\Pi}_2 (A_9) \\
+ r_3, p_1 \tilde{\Pi}_3 (A_9) - c_4, M1 \tilde{\Pi}_4 (A_9)
\end{bmatrix} + \begin{bmatrix}
\tau_1, p_2 \tilde{\Pi}_1 (A_{25}) + r_2, p_1 \tilde{\Pi}_2 (A_{25}) \\
- c_3, M1 \tilde{\Pi}_3 (A_{25}) - c_4, M1 \tilde{\Pi}_4 (A_{25})
\end{bmatrix}
\end{align*}
$$

Substituting (16), (17) and (18) into $EV (A_9 = [P_2, P_1, P_1, M1])$ provides:

$$
\begin{align*}
& \frac{r_1, p_2 (\tilde{\Pi}_1 (A_{25}) - \Delta^{M1,P1}_{1,3}) + r_2, p_1 (\tilde{\Pi}_2 (A_{25}) - \Delta^{M1,P1}_{2,3}) + r_3, p_1 \tilde{\Pi}_3 (A_{25}) - c_4, M1 (\tilde{\Pi}_4 (A_{25}) + \Delta^{M1,P1}_{4,3})}{\tau_1, p_2 (\tilde{\Pi}_1 (A_{25}) - \Delta^{M1,P1}_{1,3}) + \tau_2, p_1 (\tilde{\Pi}_2 (A_{25}) - \Delta^{M1,P1}_{2,3}) + \tau_3, p_1 \tilde{\Pi}_3 (A_{25}) + \tau_4, M1 (\tilde{\Pi}_4 (A_{25}) + \Delta^{M1,P1}_{4,3})}
\end{align*}
$$

By adding and subtracting the terms $\tau_3, M1 \tilde{\Pi}_3 (A_{25})$ in the denominator of $EV (A_9 = [P_2, P_1, P_1, M1])$ provides:

$$
\begin{align*}
& \frac{r_1, p_2 (\tilde{\Pi}_1 (A_{25}) - \Delta^{M1,P1}_{1,3}) + r_2, p_1 (\tilde{\Pi}_2 (A_{25}) - \Delta^{M1,P1}_{2,3}) + r_3, p_1 \tilde{\Pi}_3 (A_{25}) - c_4, M1 (\tilde{\Pi}_4 (A_{25}) + \Delta^{M1,P1}_{4,3})}{\tau_1, p_2 (\tilde{\Pi}_1 (A_{25}) + r_2, p_1 \tilde{\Pi}_2 (A_{25}) - c_3, M1 \tilde{\Pi}_3 (A_{25}) - c_4, M1 \tilde{\Pi}_4 (A_{25}) + \tau_3, p_1 \tilde{\Pi}_3 (A_{25}) + \tau_4, M1 \tilde{\Pi}_4 (A_{25})}
\end{align*}
$$

$$
\begin{align*}
& = \begin{bmatrix}
\tau_1, p_2 \tilde{\Pi}_1 (A_{25}) + r_2, p_1 \tilde{\Pi}_2 (A_{25}) - c_3, M1 \tilde{\Pi}_3 (A_{25}) - c_4, M1 \tilde{\Pi}_4 (A_{25}) \\
+ EV (A_{25}) (\tau_1, p_2 \Delta^{M1,P1}_{1,3} - r_2, p_1 \Delta^{M1,P1}_{2,3} + (\tau_3, p_1 - \tau_3, M1) \tilde{\Pi}_3 (A_{25}) + \tau_4, M1 \Delta^{M1,P1}_{4,3})
\end{bmatrix}
\end{align*}
$$
\[
\begin{align*}
    r_{3,p_1} \Pi_3(A_{25}) &= -c_{3,M_1} \Pi_3(A_{25}) + r_{1,p_2} \Delta_{1,3}^{M_1,P_1} + r_{2,p_1} \Delta_{2,3}^{M_1,P_1} + c_{4,M_1} \Delta_{4,3}^{M_1,P_1} \\
    + EV(A_{25}) \left\{ -r_{1,p_2} \Delta_{1,3}^{M_1,P_1} - r_{2,p_1} \Delta_{2,3}^{M_1,P_1} + (\tau_3,p_1 - \tau_3,M_1) \Pi_3(A_{25}) + r_{4,M_1} \Delta_{4,3}^{M_1,P_1} \right\} \\
    r_{2,p_1} &= -c_{3,M_1} + r_{1,p_2} \left( \frac{\Delta_{1,3}^{M_1,P_1}}{\Pi_3(A_{25})} \right) + r_{2,p_1} \left( \frac{\Delta_{2,3}^{M_1,P_1}}{\Pi_3(A_{25})} \right) + c_{4,M_1} \left( \frac{\Delta_{4,3}^{M_1,P_1}}{\Pi_3(A_{25})} \right) \\
    + EV(A_{25}) \left\{ -\tau_1,p_1 \left( \frac{\Delta_{1,3}^{M_1,P_1}}{\Pi_3(A_{25})} \right) - \tau_2,p_1 \left( \frac{\Delta_{2,3}^{M_1,P_1}}{\Pi_3(A_{25})} \right) + (\tau_3,p_1 - \tau_3,M_1) + r_{4,M_1} \left( \frac{\Delta_{4,3}^{M_1,P_1}}{\Pi_3(A_{25})} \right) \right\} \\
    \frac{r_{3,p_1}}{c_{3,M_1}} &= -1 + \left( \frac{r_{1,p_1}}{c_{3,M_1}} \right) \left( \frac{\Delta_{1,3}^{M_1,P_1}}{\Pi_3(A_{25})} \right) + \left( \frac{r_{2,p_1}}{c_{3,M_1}} \right) \left( \frac{\Delta_{2,3}^{M_1,P_1}}{\Pi_3(A_{25})} \right) + \left( \frac{c_{4,M_1}}{c_{3,M_1}} \right) \left( \frac{\Delta_{4,3}^{M_1,P_1}}{\Pi_3(A_{25})} \right) \\
    + \left( EV(A_{25}) \right) \left\{ -\tau_1,p_1 \left( \frac{\Delta_{1,3}^{M_1,P_1}}{\Pi_3(A_{25})} \right) - \tau_2,p_1 \left( \frac{\Delta_{2,3}^{M_1,P_1}}{\Pi_3(A_{25})} \right) + (\tau_3,p_1 - \tau_3,M_1) + r_{4,M_1} \left( \frac{\Delta_{4,3}^{M_1,P_1}}{\Pi_3(A_{25})} \right) \right\}
\end{align*}
\]

Note that \( EV(A_{25}) \times \left( -\tau_1,p_2 \Pi_1(A_{25}) - \tau_2,p_1 \Pi_2(A_{25}) - \tau_3,M_1 \Pi_3(A_{25}) - \tau_4,M_1 \Pi_4(A_{25}) \right) = -r_1,p_2 \Pi_1(A_{25}) - r_2,p_1 \Pi_2(A_{25}) + c_{3,M_1} \Pi_3(A_{25}) + c_{4,M_1} \Pi_4(A_{25}) \). Therefore, substituting \(-\tau_3,M_1 \Pi_3(A_{25}) EV(A_{25}) = -r_1,p_2 \Pi_1(A_{25}) - r_2,p_1 \Pi_2(A_{25}) + c_{3,M_1} \Pi_3(A_{25}) + c_{4,M_1} \Pi_4(A_{25}) + EV(A_{25}) \times \left( \tau_1,p_2 \Pi_1(A_{25}) \tau_2,p_1 \Pi_2(A_{25}) + r_4,M_1 \Pi_4(A_{25}) \right) \) into the above expression provides

\[
\begin{align*}
    \frac{r_{3,p_1}}{c_{3,M_1}} &= -1 + \left( \frac{1}{c_{3,M_1} \Pi_3(A_{25})} \right) \left[ -r_1,p_2 \left( \Pi_1(A_{25}) - \Delta_{1,3}^{M_1,P_1} \right) - r_2,p_1 \left( \Pi_2(A_{25}) - \Delta_{2,3}^{M_1,P_1} \right) \\
    + c_{3,M_1} \Pi_3(A_{25}) + c_{4,M_1} \left( \Pi_4(A_{25}) + \Delta_{4,3}^{M_1,P_1} \right) \right] \\
    + EV(A_{25}) \left\{ \tau_1,p_2 \left( \Pi_1(A_{25}) - \Delta_{1,3}^{M_1,P_1} \right) - \tau_2,p_1 \left( \Pi_2(A_{25}) - \Delta_{2,3}^{M_1,P_1} \right) + (\tau_3,p_1 - \tau_3,M_1) + r_{4,M_1} \left( \Pi_4(A_{25}) + \Delta_{4,3}^{M_1,P_1} \right) \right\}
\end{align*}
\]

\[
\begin{align*}
    \frac{r_{3,p_1}}{c_{3,M_1}} &= - \left( \frac{r_1,p_2}{c_{3,M_1}} \right) \left( \Pi_1(A_{25}) - \Delta_{1,3}^{M_1,P_1} \right) - \left( \frac{r_2,p_1}{c_{3,M_1}} \right) \left( \Pi_2(A_{25}) - \Delta_{2,3}^{M_1,P_1} \right) \\
    + \left( \frac{c_{3,M_1}}{c_{3,M_1}} \right) \left( \Pi_4(A_{25}) + \Delta_{4,3}^{M_1,P_1} \right) + EV(A_{25}) \left\{ \tau_1,p_2 \left( \Pi_1(A_{25}) - \Delta_{1,3}^{M_1,P_1} \right) - \tau_2,p_1 \left( \Pi_2(A_{25}) - \Delta_{2,3}^{M_1,P_1} \right) + (\tau_3,p_1 - \tau_3,M_1) + r_{4,M_1} \left( \Pi_4(A_{25}) + \Delta_{4,3}^{M_1,P_1} \right) \right\}
\end{align*}
\]

The critical ratio \( \gamma_{3,3}^{M_1,P_1} \) for switching from minor maintenance \( M_1 \) to low-end technology product \( P_1 \) is then expressed.
as follows:

\[
\gamma_{3,M_1}^{M_1,P_1} = \frac{r_{3,P_1}}{c_{3,M_1}} = - \left( \frac{\hat{\Pi}_1(A_{25}) - \Delta_{1,3}^{M_1,P_1}}{\Pi_1(A_{25})} \right) - \left( \frac{\hat{\Pi}_2(A_{25}) - \Delta_{2,3}^{M_1,P_1}}{\Pi_2(A_{25})} \right) + \left( \frac{\hat{\Pi}_4(A_{25}) + \Delta_{4,3}^{M_1,P_1}}{\Pi_4(A_{25})} \right) + EV(A_{25}) \left\{ \begin{array}{c}
\tau_{1,P_2} \left( \hat{\Pi}_1(A_{25}) - \Delta_{1,3}^{M_1,P_1} \right) \\
\tau_{2,P_1} \left( \hat{\Pi}_2(A_{25}) - \Delta_{2,3}^{M_1,P_1} \right) \\
+ \tau_{4,M_1} \left( \hat{\Pi}_4(A_{25}) + \Delta_{4,3}^{M_1,P_1} \right)
\end{array} \right\}.
\]

Therefore, when \( \frac{r_{3,P_1}}{c_{3,M_1}} = \gamma_{3,M_1}^{M_1,P_1} \), then the firm is indifferent between production and maintenance actions in state 3 because \( EV(A_9 = [P_2, P_1, P_1, M_1]) = EV(A_{25} = [P_2, P_1, M_1, M_1]) \). This proves part c) of the Proposition. Note that the value of \( \frac{r_{3,P_1}}{c_{3,M_1}} \) is positive by definition, and therefore, if \( \gamma_{3,M_1}^{M_1,P_1} < 0 \), it implies that \( EV(A_9 = [P_2, P_1, P_1, M_1]) > EV(A_{25} = [P_2, P_1, M_1, M_1]) \), and therefore \( a_3^* = P_1 \). a) When \( \frac{r_{3,P_1}}{c_{3,M_1}} > \gamma_{3,M_1}^{M_1,P_1} \), the firm has \( EV(A_9 = [P_2, P_1, P_1, M_1]) > EV(A_{25} = [P_2, P_1, M_1, M_1]) \), and therefore \( a_3^* = P_1 \). b) When \( \frac{r_{3,P_1}}{c_{3,M_1}} < \gamma_{3,M_1}^{M_1,P_1} \), the firm has \( EV(A_9 = [P_2, P_1, P_1, M_1]) < EV(A_{25} = [P_2, P_1, M_1, M_1]) \) and therefore \( a_3^* = M_1 \).

**Proof of Proposition 2:** We first derive the critical ratio expression by equating the expected values of the two policies \( A_9 = [P_2, P_1, P_1, M_1] \) and \( A_{11} = [P_2, P_1, P_2, M_1] \). Recall that

\[
\hat{\Pi}_1(A_{11}) = \hat{\Pi}_1(A_9) \times \delta_{3}^{P_1,P_2}, \quad (19)
\]
\[
\hat{\Pi}_2(A_{11}) = \hat{\Pi}_2(A_9) \times \delta_{3}^{P_1,P_2}, \quad (20)
\]
\[
\hat{\Pi}_3(A_{11}) = \hat{\Pi}_3(A_9), \quad (21)
\]
\[
\hat{\Pi}_4(A_{11}) = \hat{\Pi}_4(A_9) \times \delta_{3}^{P_1,P_2}. \quad (22)
\]

Using these relationships, we can express these two policies as follows:

\[
EV(A_{11} = [P_2, P_1, P_2, M_1]) = \begin{bmatrix}
\tau_{1,P_2} \hat{\Pi}_1(A_9) + r_{2,P_1} \hat{\Pi}_2(A_{11}) \\
+ r_{3,P_2} \hat{\Pi}_3(A_{11}) - c_{4,M_1} \hat{\Pi}_4(A_{11}) \\
\end{bmatrix} = \begin{bmatrix}
\tau_{1,P_2} \hat{\Pi}_1(A_9) + r_{2,P_1} \hat{\Pi}_2(A_9) \\
+ r_{3,P_2} \hat{\Pi}_3(A_9) - c_{4,M_1} \hat{\Pi}_4(A_9)
\end{bmatrix}
\]

Substituting (19)–(22) into \( EV(A_{11} = [P_2, P_1, P_2, M_1]) \) provides the following:

\[
\begin{align*}
& = \frac{r_{1,P_2} \left( \hat{\Pi}_1(A_9) \delta_{3}^{P_1,P_2} \right) + r_{2,P_1} \left( \hat{\Pi}_2(A_9) \delta_{3}^{P_1,P_2} \right) + r_{3,P_2} \hat{\Pi}_3(A_9) - c_{4,M_1} \left( \hat{\Pi}_4(A_9) \delta_{3}^{P_1,P_2} \right)}{\tau_{1,P_1} \left( \hat{\Pi}_1(A_9) \delta_{3}^{P_1,P_2} \right) + r_{2,P_2} \left( \hat{\Pi}_2(A_9) \delta_{3}^{P_1,P_2} \right) + r_{3,P_2} \hat{\Pi}_3(A_9) + \tau_{4,M_1} \left( \hat{\Pi}_4(A_9) \delta_{3}^{P_1,P_2} \right)} \\
& = \frac{r_{1,P_2} \hat{\Pi}_1(A_9) + r_{2,P_1} \hat{\Pi}_2(A_9) + r_{3,P_1} \hat{\Pi}_3(A_9) - c_{4,M_1} \hat{\Pi}_4(A_9)}{\tau_{1,P_1} \hat{\Pi}_1(A_9) + r_{2,P_2} \hat{\Pi}_2(A_9) + r_{3,P_1} \hat{\Pi}_3(A_9) + \tau_{4,M_1} \hat{\Pi}_4(A_9)}
\end{align*}
\]
Adding and subtracting $\tau_3, p_1 \tilde{\Pi}_3 (A_9) \delta_3^{P_1, P_2}$ into the denominator of $EV (A_{11} = [P_2, P_1, P_2, M_1])$ provides:

$$
\begin{align*}
& r_1, p_2 (\tilde{\Pi}_1 (A_9) \delta_3^{P_1, P_2}) + r_2, p_1 (\tilde{\Pi}_2 (A_9) \delta_3^{P_1, P_2}) + r_3, p_2 \tilde{\Pi}_3 (A_9) - c_4, M_1 (\tilde{\Pi}_4 (A_9) \delta_3^{P_1, P_2}) \\
& = r_1, p_2 (\tilde{\Pi}_1 (A_9) \delta_3^{P_1, P_2}) + r_2, p_1 (\tilde{\Pi}_2 (A_9) \delta_3^{P_1, P_2}) + r_3, p_1 (\tilde{\Pi}_3 (A_9) \delta_3^{P_1, P_2}) - c_4, M_1 (\tilde{\Pi}_4 (A_9) \delta_3^{P_1, P_2}) \\
& + EV (A_9) \tilde{\Pi}_3 (A_9) \left\{ \tau_3, p_2 - \tau_3, p_1 \delta_3^{P_1, P_2} \right\} \\
& \quad - \tau_3, p_2 \tilde{\Pi}_3 (A_9) = r_3, p_1 (\tilde{\Pi}_3 (A_9) \delta_3^{P_1, P_2}) + EV (A_9) \tilde{\Pi}_3 (A_9) \left( \tau_3, p_2 - \tau_3, p_1 \delta_3^{P_1, P_2} \right) \\
& \alpha_3^{P_1, P_2} = \frac{r_3, p_2}{r_3, p_1} \tau_3, p_1 \delta_3^{P_1, P_2} + EV (A_9) \left( \frac{\tau_3, p_2 - \tau_3, p_1 \delta_3^{P_1, P_2}}{r_3, p_1} \right).
\end{align*}
$$

Because product $P_2$ earns a higher profit than $P_1$ in each state, $r_3, p_2 > r_3, p_1$, and therefore the firm has $\frac{r_3, p_2}{r_3, p_1} > 1$.

If $\alpha_3^{P_1, P_2} \leq 1$, it implies that $EV (A_{11} = [P_2, P_1, P_2, M_1]) \geq EV (A_9 = [P_2, P_1, P_1, M_1])$ and $\alpha_3^* = P_2$. However, if $\alpha_3^{P_1, P_2} > 1$, then the firm needs to compare the ratio of profits earned from producing $P_1$ and $P_2$ in state 3, $\frac{r_3, p_2}{r_3, p_1}$, with the critical ratio $\alpha_3^{P_1, P_2}$. a) When $\frac{r_3, p_2}{r_3, p_1} > \alpha_3^{P_1, P_2}$, then $EV (A_{11} = [P_2, P_1, P_2, M_1]) > EV (A_9 = [P_2, P_1, P_1, M_1])$ and $\alpha_3^* = P_2$. b) If $\frac{r_3, p_2}{r_3, p_1} < \alpha_3^{P_1, P_2}$, then $EV (A_{11} = [P_2, P_1, P_2, M_1]) < EV (A_9 = [P_2, P_1, P_1, M_1])$ and $\alpha_3^* = P_1$. c) When $\frac{r_3, p_2}{r_3, p_1} = \alpha_3^{P_1, P_2}$, then $EV (A_{11} = [P_2, P_1, P_2, M_1]) = EV (A_9 = [P_2, P_1, P_1, M_1])$ and the firm is indifferent between producing $P_1$ or $P_2$ in state 3.

**Proof of Proposition 3:** We first derive the critical ratio expression by equating the expected values of the two policies $A_{25} = [P_2, P_1, M_1, M_1]$ and $A_{27} = [P_2, P_1, M_2, M_1]$. Recall that

$$
\begin{align*}
\tilde{\Pi}_1 (A_{27}) &= \tilde{\Pi}_1 (A_{25}) \times \delta_3^{M_1, M_2}, \\
\tilde{\Pi}_2 (A_{27}) &= \tilde{\Pi}_2 (A_{25}) \times \delta_3^{M_1, M_2}, \\
\tilde{\Pi}_3 (A_{27}) &= \tilde{\Pi}_3 (A_{25}), \\
\tilde{\Pi}_4 (A_{27}) &= \tilde{\Pi}_4 (A_{25}) \times \delta_3^{M_1, M_2}.
\end{align*}
$$

Using these relationships, the policies can be expressed as follows:

$$
\begin{align*}
EV (A_{27} = [P_2, P_1, M_2, M_1]) &= EV (A_{25} = [P_2, P_1, M_1, M_1]) \\
\left[ \begin{array}{c}
\tau_1, p_2 \tilde{\Pi}_1 (A_{27}) + \tau_2, p_1 \tilde{\Pi}_2 (A_{27}) \\
- c_4, M_2 \tilde{\Pi}_4 (A_{27}) - c_4, M_1 \tilde{\Pi}_4 (A_{27})
\end{array} \right] &= \left[ \begin{array}{c}
\tau_1, p_2 \tilde{\Pi}_1 (A_{25}) + \tau_2, p_1 \tilde{\Pi}_2 (A_{25}) \\
- c_4, M_2 \tilde{\Pi}_4 (A_{25}) - c_4, M_1 \tilde{\Pi}_4 (A_{25})
\end{array} \right].
\end{align*}
$$

Using these relationships, the policies can be expressed as follows:
Substituting Using (23)–(26) into $EV (A_{27} = \{P_2, P_1, M_2, M_1\})$ provides the following:

\[ r_{1,2} \tilde{\Pi}_1 (A_{25}) \delta_3^{M_1,M_2} + r_{2,1} \tilde{\Pi}_2 (A_{25}) \delta_3^{M_1,M_2} - c_{3,M_2} \tilde{\Pi}_3 (A_{25}) - c_{4,M_1} \tilde{\Pi}_4 (A_{25}) \delta_3^{M_1,M_2} \\
\] 
\[ r_{1,2} \tilde{\Pi}_1 (A_{25}) \delta_3^{M_1,M_2} + r_{2,1} \tilde{\Pi}_2 (A_{25}) \delta_3^{M_1,M_2} + \tau_{3,1} \tilde{\Pi}_1 (A_{25}) + \tau_{4,1} \tilde{\Pi}_2 (A_{25}) \delta_3^{M_1,M_2} \]

Adding and subtracting $\tau_{3,1} \tilde{\Pi}_3 (A_{25}) \delta_3^{M_1,M_2}$ into the denominator of $EV (A_{27} = \{P_2, P_1, M_2, M_1\})$ provides:

\[ r_{1,2} \tilde{\Pi}_1 (A_{25}) \delta_3^{M_1,M_2} + r_{2,1} \tilde{\Pi}_2 (A_{25}) \delta_3^{M_1,M_2} - c_{3,M_2} \tilde{\Pi}_3 (A_{25}) - c_{4,M_1} \tilde{\Pi}_4 (A_{25}) \delta_3^{M_1,M_2} + EV (A_{25}) \tilde{\Pi}_3 (A_{25}) \left( \tau_{3,2} - \tau_{3,1} \delta_3^{M_1,M_2} \right) \]

Because the cost of the maintenance action $M_2$ is higher than that of $M_1$, i.e., $c_{3,M_2} > c_{3,M_1}$, the firm has $c_{3,M_2} > c_{3,M_1}$. If $\lambda_3^{M_1,M_2} < 1$, it implies that $EV (A_{25} = \{P_2, P_1, M_1, M_1\}) \geq EV (A_{27} = \{P_2, P_1, M_2, M_1\})$ and $a_3 = M_1$. However, if $\lambda_3^{M_1,M_2} > 1$, then the firm needs to compare the ratio of the maintenance costs stemming from $M_1$ and $M_2$. $c_{3,M_2} > c_{3,M_1}$ with the critical ratio $\lambda_3^{M_1,M_2}$. a) If $c_{3,M_2} > \lambda_3^{M_1,M_2}$, then $EV (A_{25} = \{P_2, P_1, M_1, M_1\}) > EV (A_{27} = \{P_2, P_1, M_2, M_1\})$ and $a_3 = M_1$. b) If $c_{3,M_2} < \lambda_3^{M_1,M_2}$, then $EV (A_{25} = \{P_2, P_1, M_1, M_1\}) < EV (A_{27} = \{P_2, P_1, M_2, M_1\})$ and $a_3 = M_2$. c) If $c_{3,M_2} > \lambda_3^{M_1,M_2}$, then $EV (A_{25} = \{P_2, P_1, M_1, M_1\}) = EV (A_{27} = \{P_2, P_1, M_2, M_1\})$ and the firm is indifferent between the maintenance actions $M_1$ and $M_2$ in state 3.

**Proof of Proposition 4:** Before we proceed with the proof, let us first establish how the combination of the critical ratios can determine the decision-maker’s preferences.

1. If $r_{3,4} > \frac{\gamma_3^{M_1,P_1} \times \alpha_3^{P_1,P_2}}{M_2}$, then the policy $A_{11} = \{P_2, P_1, P_2, M_1\}$ is preferred over the policy $A_{25} = \{P_2, P_1, M_1, M_1\}$; otherwise, if $r_{3,4} < \frac{\gamma_3^{M_1,P_1} \times \alpha_3^{P_1,P_2}}{M_2}$, then the policy $A_{25} = \{P_2, P_1, M_1, M_1\}$ is preferred. The proof of this statement follows from Propositions 1 and 2.

2. If $r_{3,4} > \frac{\gamma_3^{M_1,P_1} \times \alpha_3^{P_1,P_2}}{M_2}$, then the policy $A_{11} = \{P_2, P_1, M_2, M_1\}$ is preferred over the policy $A_{27} = \{P_2, P_1, M_2, M_1\}$; otherwise, if $r_{3,4} < \frac{\gamma_3^{M_1,P_1} \times \alpha_3^{P_1,P_2}}{M_2}$, then the policy $A_{27} = \{P_2, P_1, M_2, M_1\}$ is preferred. The proof of this statement follows from Propositions 1 and 3.

3. If $r_{3,4} > \frac{\gamma_3^{M_1,P_1} \times \alpha_3^{P_1,P_2}}{M_2}$, then the policy $A_{11} = \{P_2, P_1, M_2, M_1\}$ is preferred over the policy $A_{27} = \{P_2, P_1, M_2, M_1\}$; otherwise, if $r_{3,4} < \frac{\gamma_3^{M_1,P_1} \times \alpha_3^{P_1,P_2}}{M_2}$, then the policy $A_{27} = \{P_2, P_1, M_2, M_1\}$ is preferred. This statement follows from Propositions 1, 2 and 3.

The above three statements provide the preferences among these four policies. Using the combination of critical ratios, the firm can then determine the optimal policy by comparing the profit earned by manufacturing actions with
the associated maintenance costs.

a) From Proposition 1, when $r_{3,p2} > \alpha_3^{P1,P2}$, or when $r_{3,p1} > \frac{r_{3,p2}}{\alpha_3^{P1,P2}}$, $EV(A_9) = [P2, P1, P1, M1]) > EV(A_{11} = [P2, P1, P2, M1])$. From Proposition 1, when $r_{3,p1} > c_3M1_{\lambda_3^{P1,P2}}$, $EV(A_9) = [P2, P1, P1, M1]) > EV(A_{25} = [P2, P1, M1, M1])$. From Proposition 3, when $\frac{c_3M2_{\lambda_3^{M1,M2}}}{\lambda_3^{M1,M2}} \leq \lambda_3^{M1,M2}$, we have $EV(A_{25} = [P2, P1, M1, M1]) = EV(A_{27} = [P2, P1, M2, M1])$; therefore, when $r_{3,p1} > c_3M3_{\lambda_3^{M1,M2}}$, $EV(A_9) = [P2, P1, P1, M1]) > EV(A_{27} = [P2, P1, M2, M1])$.

Thus, $A_9 = [P2, P1, P1, M1]$ is the best policy when $r_{3,p1} \geq \left\{ r_{3,p2} > \frac{r_{3,p2}}{\alpha_3^{P1,P2}}, c_3M1_{\lambda_3^{P1,P2}}, \frac{c_3M1_{\lambda_3^{M1,P1}}}{\lambda_3^{M1,P1}}, \frac{c_3M3_{\lambda_3^{M1,P1}}}{\lambda_3^{M1,P1}} \right\}$.

b) From Proposition 2, when $\frac{r_{3,p1}}{\gamma_3^{P1,P2}} > \alpha_3^{P1,P2}$, or when $r_{3,p2} > r_{3,p1}\alpha_3^{P1,P2}$, $EV(A_{11} = [P2, P1, P2, M1]) > EV(A_9 = [P2, P1, P1, M1])$. From Proposition 1, when $r_{3,p1} = c_3M1_{\lambda_3^{P1,P2}}$, $EV(A_9) = [P2, P1, P1, M1]) = EV(A_{25} = [P2, P1, M1, M1])$. From Proposition 3, when $\frac{c_3M2_{\lambda_3^{M1,M2}}}{\lambda_3^{M1,M2}} \leq \lambda_3^{M1,M2}$, we have $EV(A_{25} = [P2, P1, M1, M1]) = EV(A_{27} = [P2, P1, M2, M1])$; therefore, when $r_{3,p2} > c_3M3_{\lambda_3^{M1,M2}}$, $EV(A_9) = [P2, P1, P1, M1]) > EV(A_{27} = [P2, P1, M2, M1])$.

Thus, $A_{11} = [P2, P1, P2, M1]$ is the best policy when $r_{3,p2} \geq \left\{ r_{3,p1}\alpha_3^{P1,P2}, c_3M1_{\lambda_3^{M1,P1}}, \frac{c_3M1_{\lambda_3^{M1,P1}}}{\lambda_3^{M1,P1}}, c_3M3_{\lambda_3^{M1,M2}} \right\}$.

c) From Proposition 1, when $c_3M1 > \frac{r_{3,p1}}{\gamma_3^{P1,P2}}$, $EV(A_{25} = [P2, P1, M1, M1]) > EV(A_9 = [P2, P1, P1, M1])$. From Proposition 2, when $r_{3,p2} > c_3M2_{\lambda_3^{M1,M2}}$, $EV(A_{25} = [P2, P1, M1, M1]) = EV(A_{11} = [P2, P1, P2, M1])$; therefore, when $c_3M1 > \frac{r_{3,p2}}{c_3M2_{\lambda_3^{M1,M2}}}$, $EV(A_{25} = [P2, P1, M1, M1]) > EV(A_{27} = [P2, P1, M2, M1])$.

Thus, the best policy is $A_{25} = [P2, P1, M1, M1]$ when $c_3M1 \geq \left\{ r_{3,p1}, r_{3,p2} > \frac{1}{\gamma_3^{P1,P2}}, r_{3,p2} > \frac{1}{\lambda_3^{M1,M2}} \right\}$ and $c_3M1 \leq c_3M2 \lambda_3^{M1,M2}$.

d) From Proposition 3, when $c_3M2 < c_3M1\lambda_3^{M1,M2}$, we have $EV(A_{27} = [P2, P1, M2, M1]) > EV(A_{25} = [P2, P1, M1, M1])$. From Proposition 1, when $c_3M1 > \frac{r_{3,p1}}{\gamma_3^{P1,P2}}$, $EV(A_{25} = [P2, P1, M1, M1]) = EV(A_9 = [P2, P1, P1, M1])$; therefore, when $c_3M2 > r_{3,p1}\lambda_3^{M1,M2}$, $EV(A_{27} = [P2, P1, M2, M1]) > EV(A_{25} = [P2, P1, M1, M1])$. From Proposition 2, when $r_{3,p2} > c_3M2_{\lambda_3^{M1,M2}}$, $EV(A_{25} = [P2, P1, M1, M1]) = EV(A_{11} = [P2, P1, P2, M1])$; therefore, when $c_3M2 > r_{3,p2} \lambda_3^{M1,M2}$, $EV(A_{27} = [P2, P1, M2, M1]) > EV(A_{25} = [P2, P1, M1, M1])$.

Thus, the best policy is $A_{27} = [P2, P1, M2, M1]$ when $c_3M2 \geq \left\{ r_{3,p1}, r_{3,p2} > \frac{1}{\gamma_3^{P1,P2}}, r_{3,p2} > \frac{1}{\lambda_3^{M1,M2}} \right\}$ and $c_3M2 \leq c_3M1\lambda_3^{M1,M2}$.

Proof of Proposition 5: a) The comparison of the expected production from policies $A_{25} = [P2, P1, M1, M1]$ and $A_9 = [P2, P1, P1, M1]$ provides the result.

$$Y_{P1}(A_9) = Y_{P1}(A_{25})$$

$$y_{2,p1}\Pi_2(A_9) + y_{3,p1}\Pi_3(A_9) \quad > \quad y_{2,p1}\Pi_2(A_{25})$$

$$y_{2,p1}\Pi_2(A_9) + y_{3,p1}\Pi_3(A_9) \quad > \quad y_{2,p1}\Pi_2(A_{25})$$
Substituting (16), (17) and (18) into $Y_{P1}(A_9)$ provides:

$$y_{2, P1} \left( \hat{\Pi}_2 (A_{25}) - \Delta_{2, 3}^{M_1, P_1} \right) + y_{3, P1} \hat{\Pi}_3 (A_{25})$$

By adding and subtracting the terms $\tau_{3, M_1} \hat{\Pi}_3 (A_{25})$ in the denominator of $Y_{P1}(A_9)$ provides:

$$y_{2, P1} \left( \hat{\Pi}_2 (A_{25}) - \Delta_{2, 3}^{M_1, P_1} \right) + y_{3, P1} \hat{\Pi}_3 (A_{25})$$

b) The comparison of the expected production from policies $A_{25} = [P2, P1, M1, M1]$ and $A_{27} = [P2, P1, M2, M1]$ provides the result.

$$Y_{P1}(A_{27}) > Y_{P1}(A_{25})$$

Substituting Using (23)–(26) into $Y_{P1}(A_{27})$ provides the following:

$$y_{2, P1} \hat{\Pi}_2 (A_{25}) > \frac{\tau_{1, P2} \hat{\Pi}_1 (A_{27}) + \tau_{2, P1} \hat{\Pi}_2 (A_{27})}{\tau_{1, P2} \hat{\Pi}_1 (A_{25}) + \tau_{2, P1} \hat{\Pi}_2 (A_{25}) + \tau_{3, M1} \hat{\Pi}_3 (A_{27}) + \tau_{4, M1} \hat{\Pi}_4 (A_{27})}$$

Adding and subtracting $\tau_{3, M1} \hat{\Pi}_3 (A_{25}) \delta_3^{M_1, M_2}$ into the denominator of $Y_{P1}(A_{27})$ provides:

$$y_{2, P1} \hat{\Pi}_2 (A_{25}) \delta_3^{M_1, M_2} > y_{2, P1} \hat{\Pi}_2 (A_{25}) \delta_3^{M_1, M_2} + y_{P1} (A_{25}) \hat{\Pi}_3 (A_{25}) \left( \tau_{3, M_2} - \tau_{3, M_1} \delta_3^{M_1, M_2} \right)$$

$$Y_{P1}(A_{27}) \hat{\Pi}_3 (A_{25}) \left( \tau_{3, M_2} - \tau_{3, M_1} \delta_3^{M_1, M_2} \right) < 0.$$
Because \( Y_{p1}(A_{25}) \) and \( \hat{\Pi}_3(A_{25}) \) are both positive, the above is satisfied when

\[
\left( \tau_{3,M2} - \tau_{3,M1} \delta_{3,1}^{M1,M2} \right) < 0.
\]

A similar condition can be obtained for \( P2 \).

c) The comparison of the expected production from policies \( A_{25} = [P2, P1, M1, M1] \) and \( A_{41} = [P2, M1, P2, M1] \) provides the result.

\[
Y_{p2}(A_{41}) > Y_{p2}(A_{25})
\]

Replicating (16)–(18) for state 2 and then substituting the relationships from states 2 and 3 into \( Y_{p2}(A_{41}) \) provide the following:

\[
\frac{y_{1,2} \Pi_1(A_{25}) + \Delta_{1,2}^{M1,P1} - \Delta_{1,3}^{M1,P1}}{\tau_{1,2} \Pi_1(A_{25}) + \Pi_2(A_{25}) + \tau_{3,1} \Pi_3(A_{25}) + \tau_{4,1} \Pi_4(A_{25})}
\]

Adding and subtracting \( \tau_{2,1} \Pi_2(A_{25}) + \tau_{3,1} \Pi_3(A_{25}) \) into the denominator of \( Y_{p2}(A_{41}) \) provides:

\[
y_{1,2} \Pi_1(A_{25}) + \Delta_{1,2}^{M1,P1} - \Delta_{1,3}^{M1,P1} > y_{1,2} \Pi_2(A_{25})
\]

A similar condition can be obtained for \( P1 \).

Proof of Proposition 6: The proof follows from the comparison of the numerator terms in steady-state probabilities in (2)–(5). a) In Group 3 policies, product \( P1 \) is manufactured only in state 3 in policies \( A_{41}, A_{42}, A_{45} \), and \( A_{46} \). In
these four policies, the expected production of $P_1$ is equal to:

$$Y_{P1}(A) = \frac{y_3, P_1 \Pi_3(A)}{\sum_{i=1}^{N} \tau_{i,a} \Pi_i(A)} = \frac{y_3, P_1 \hat{\Pi}_3(A)}{\sum_{i=1}^{N} \tau_{i,a} \hat{\Pi}_i(A)}.$$  

From (4), it can be observed that $\hat{\Pi}_3(A_{42}) \geq \left\{ \hat{\Pi}_3(A_{41}), \hat{\Pi}_3(A_{43}), \hat{\Pi}_3(A_{46}) \right\}$. Thus, $Y_{P1}(A_{42}) \geq \{ Y_{P1}(A_{41}), Y_{P1}(A_{43}), Y_{P1}(A_{46}) \}$, resulting in the highest expected production amount among these four policies. Then, if $MPR_{P1} > Y_{P1}(A_{42})$, these four policies cannot be optimal. Similarly, product $P_2$ is manufactured only in state 3 in policies $A_{35}$, $A_{36}$, $A_{39}$, and $A_{40}$. In these four policies, the expected production of $P_2$ is equal to:

$$Y_{P2}(A) = \frac{y_3, P_2 \Pi_3(A)}{\sum_{i=1}^{N} \tau_{i,a} \Pi_i(A)} = \frac{y_3, P_2 \hat{\Pi}_3(A)}{\sum_{i=1}^{N} \tau_{i,a} \hat{\Pi}_i(A)}.$$  

From (4), it can be observed that $\hat{\Pi}_3(A_{36}) \geq \left\{ \hat{\Pi}_3(A_{35}), \hat{\Pi}_3(A_{39}), \hat{\Pi}_3(A_{40}) \right\}$. Thus, $Y_{P2}(A_{36}) \geq \{ Y_{P2}(A_{35}), Y_{P2}(A_{39}), Y_{P2}(A_{40}) \}$, resulting in the highest expected production amount among these four policies. Then, if $MPR_{P2} > Y_{P2}(A_{36})$, these four policies cannot be optimal. Collectively, the conditions $MPR_{P1} > Y_{P1}(A_{42})$ and $MPR_{P2} > Y_{P2}(A_{36})$ eliminate all eight policies in Group 3 from the list of potentially optimal policies, reducing its set to 20 policies. The consequence of eliminating Group 3 policies is that the firm has to commit to production in the first two states. As a result, in state 2, the decision maker does not have to use the two sets of critical ratios $\alpha_{1, P1}^{M1,M2}$ and $\alpha_{1, P1}^{M1,M2}$. Like state 1, in state 2, the only comparison should be made using the critical ratio $\alpha_{1, P1}^{P1,P2}$ in order to determine which product should be manufactured.

b) Among Group 2 policies, product $P_1$ is manufactured in state 1 in policies $A_{21}$ through $A_{24}$. In these four policies, the expected production of $P_1$ is equal to:

$$Y_{P1}(A) = \frac{y_1, P_1 \Pi_1(A)}{\sum_{i=1}^{N} \tau_{i,a} \Pi_i(A)} = \frac{y_1, P_1 \hat{\Pi}_1(A)}{\sum_{i=1}^{N} \tau_{i,a} \hat{\Pi}_i(A)}.$$  

From (2), it can be observed that $\hat{\Pi}_1(A_{24}) \geq \left\{ \hat{\Pi}_1(A_{21}), \hat{\Pi}_1(A_{22}), \hat{\Pi}_1(A_{23}) \right\}$. Thus, $Y_{P1}(A_{24}) \geq \{ Y_{P1}(A_{21}), Y_{P1}(A_{22}), Y_{P1}(A_{23}) \}$, resulting in the highest expected production amount among these four policies. Then, if $MPR_{P1} > Y_{P1}(A_{24})$, these four policies cannot be optimal. Similarly, product $P_2$ is manufactured only in state 1 in policies $A_{25}$ through $A_{28}$. In these four policies, the expected production of $P_2$ is equal to:

$$Y_{P2}(A) = \frac{y_1, P_2 \Pi_1(A)}{\sum_{i=1}^{N} \tau_{i,a} \Pi_i(A)} = \frac{y_1, P_2 \hat{\Pi}_1(A)}{\sum_{i=1}^{N} \tau_{i,a} \hat{\Pi}_i(A)}.$$  

From (2), it can be observed that $\hat{\Pi}_1(A_{28}) \geq \left\{ \hat{\Pi}_1(A_{25}), \hat{\Pi}_1(A_{26}), \hat{\Pi}_1(A_{27}) \right\}$. Thus, $Y_{P2}(A_{28}) \geq \{ Y_{P2}(A_{25}), Y_{P2}(A_{26}), Y_{P2}(A_{27}) \}$, resulting in the highest expected production amount among these four policies. Then, if $MPR_{P2} > Y_{P2}(A_{28})$, these four policies cannot be optimal. Collectively, the conditions $MPR_{P1} > Y_{P1}(A_{24})$ and $MPR_{P2} > Y_{P2}(A_{28})$ eliminate all eight policies in Group 2 from the list of potentially optimal policies, reducing its set to 20 policies. The consequence of eliminating Group 2 policies is that the firm has to commit to production in the third state. As a result, in state 3, the decision maker does not have to use the two sets of critical ratios $\alpha_{1, P1}^{M1,M2}$ and $\alpha_{1, P1}^{M1,M2}$. Like state 1, in state 3, the only comparison should be made using the critical ratio $\alpha_{1, P1}^{P1,P2}$ in order to
c) In Group 3 policies, product P1 is manufactured only in state 1 in policies A_{35}, A_{36}, A_{39}, and A_{40}. In these four policies, the expected production of P1 is equal to:

\[ Y_{P1}(A) = y_{1,P1} \Pi_1(A) / \sum_{i=1}^{N} \tau_{i,a} \Pi_i(A) = y_{1,P1} \hat{\Pi}_1(A) / \sum_{i=1}^{N} \tau_{i,a} \hat{\Pi}_i(A). \]

From (2), it can be observed that \( \hat{\Pi}_1(A_{40}) \geq \left\{ \hat{\Pi}_1(A_{35}), \hat{\Pi}_1(A_{36}), \hat{\Pi}_1(A_{39}) \right\}. \) Moreover, again from (2), it can be verified that \( \hat{\Pi}_1(A_{40}) \geq \left\{ \hat{\Pi}_3(A_{41}), \hat{\Pi}_3(A_{42}), \hat{\Pi}_3(A_{43}), \hat{\Pi}_3(A_{46}) \right\}. \) It is already known that \( y_{1,P1} > y_{3,P1}; \) thus, \( Y_{P1}(A_{40}) \geq \{ Y_{P1}(A_{35}), Y_{P1}(A_{36}), Y_{P1}(A_{39}), Y_{P1}(A_{41}), Y_{P1}(A_{42}), Y_{P1}(A_{43}), Y_{P1}(A_{45}), Y_{P1}(A_{46}) \}. \) From the comparison of (2) among policies in Groups 2 and 3 and from the analysis in part b, it can be observed that \( \hat{\Pi}_1(A_{40}) \geq \left\{ \hat{\Pi}_1(A_{21}), \hat{\Pi}_1(A_{22}), \hat{\Pi}_1(A_{23}) \right\} \geq \{ \hat{\Pi}_2(A_{25}), \hat{\Pi}_2(A_{26}), \hat{\Pi}_2(A_{27}), \hat{\Pi}_2(A_{28}) \}; \) thus, \( Y_{P1}(A_{40}) \geq \{ Y_{P1}(A_{21}), Y_{P1}(A_{22}), Y_{P1}(A_{23}), Y_{P1}(A_{24}), Y_{P1}(A_{25}), Y_{P1}(A_{26}), Y_{P1}(A_{27}), Y_{P1}(A_{28}) \}. \) This implies that the expected production of P1 is maximized in policy A_{40} among all policies in Groups 2 and 3. Then, if \( MPR_{P1} > Y_{P1}(A_{40}) \), no policy in Groups 2 and 3 can be optimal, reducing the set of potentially optimal policies to those in Group 1 (with 12 policies). The consequence of this result is that manufacturing has to take place in states 1, 2, and 3. As a result, the decision maker can eliminate the critical ratios \( \gamma_2^{M1,P1}, \gamma_2^{M1,M2}, \gamma_3^{M1,P1}, \) and \( \lambda_4^{M1,M2}. \) The decision is now limited to the comparison of the production-related critical ratios \( \alpha_1^{P1,P2}, \alpha_2^{P1,P2}, \) and \( \alpha_3^{P1,P2} \) in states 1, 2, and 3, and \( \lambda_4^{M1,M2} \) in state 4.

Similarly, product P2 is manufactured only in state 1 in Group 3 policies A_{41}, A_{42}, A_{45}, and A_{46}. In these four policies, the expected production of P2 is equal to:

\[ Y_{P2}(A) = y_{1,P2} \Pi_1(A) / \sum_{i=1}^{N} \tau_{i,a} \Pi_i(A) = y_{1,P2} \hat{\Pi}_1(A) / \sum_{i=1}^{N} \tau_{i,a} \hat{\Pi}_i(A). \]

From (2), it can be observed that \( \hat{\Pi}_1(A_{46}) \geq \left\{ \hat{\Pi}_1(A_{41}), \hat{\Pi}_1(A_{42}), \hat{\Pi}_1(A_{45}) \right\}. \) Moreover, again from (2), it can be verified that \( \hat{\Pi}_1(A_{46}) \geq \left\{ \hat{\Pi}_3(A_{45}), \hat{\Pi}_3(A_{46}) \right\}. \) It is already known that \( y_{1,P2} > y_{3,P2}; \) thus, \( Y_{P2}(A_{46}) \geq \{ Y_{P2}(A_{35}), Y_{P2}(A_{36}), Y_{P2}(A_{39}), Y_{P2}(A_{40}), Y_{P2}(A_{41}), Y_{P2}(A_{42}), Y_{P2}(A_{43}), Y_{P2}(A_{45}) \}. \) From the comparison of (2) among policies in Groups 2 and 3 and from the analysis in part b, it can be observed that \( \hat{\Pi}_1(A_{46}) \geq \hat{\Pi}_1(A_{28}) \geq \left\{ \hat{\Pi}_2(A_{25}), \hat{\Pi}_2(A_{26}), \hat{\Pi}_2(A_{27}), \hat{\Pi}_2(A_{28}) \right\}; \) thus, \( Y_{P2}(A_{46}) \geq \{ Y_{P2}(A_{21}), Y_{P2}(A_{22}), Y_{P2}(A_{23}), Y_{P2}(A_{24}), Y_{P2}(A_{25}), Y_{P2}(A_{26}), Y_{P2}(A_{27}), Y_{P2}(A_{28}) \}. \) This implies that the expected production of P2 is maximized in policy A_{46} among all policies in Groups 2 and 3. Then, if \( MPR_{P2} > Y_{P2}(A_{46}) \), no policy in Groups 2 and 3 can be optimal, reducing the set of potentially optimal policies to those in Group 1 (with 12 policies). The consequence of this result is that manufacturing has to take place in states 1, 2, and 3. As a result, the decision maker can eliminate the critical ratios \( \gamma_2^{M1,P1}, \gamma_2^{M1,M2}, \gamma_3^{M1,P1}, \) and \( \lambda_4^{M1,M2}. \) The decision is now limited to the comparison of the production-related critical ratios \( \alpha_1^{P1,P2}, \alpha_2^{P1,P2}, \) and \( \alpha_3^{P1,P2} \) in states 1, 2, and 3, and \( \lambda_4^{M1,M2} \) in state 4.

d) From (2), it can be observed that \( \hat{\Pi}_1(A_{40}) \geq \{ \hat{\Pi}_1(A_3), \hat{\Pi}_1(A_4), \hat{\Pi}_1(A_5), \hat{\Pi}_1(A_6), \hat{\Pi}_1(A_7), \hat{\Pi}_1(A_8) \} \) and therefore, \( Y_{P1}(A_{40}) \geq \{ Y_{P1}(A_3), Y_{P1}(A_4), Y_{P1}(A_5), Y_{P1}(A_6), Y_{P1}(A_7), Y_{P1}(A_8) \}. \) As a result, when \( MPR_{P1} > Y_{P1}(A_{40}) \) policies A_3 through A_8 cannot be optimal. The decision maker can eliminate the critical ratios \( \alpha_1^{P1,P2}, \gamma_2^{M1,P1}, \gamma_2^{M1,M2}, \gamma_3^{M1,P1}, \) and \( \lambda_3^{M1,M2} \) from consideration. The decision is now limited to the comparison of the critical
ratios $\alpha_2^{P1,P2}$, and $\alpha_3^{P1,P2}$ in states 2, and 3, and $\lambda_4^{M1,M2}$ in state 4.

Similarly, it can be observed from (2), that $\tilde{\Pi}_1(A_{46}) = \{\tilde{\Pi}_1(A_{9}), \tilde{\Pi}_1(A_{10}), \tilde{\Pi}_1(A_{11}), \tilde{\Pi}_1(A_{12}), \tilde{\Pi}_1(A_{13}), \tilde{\Pi}_1(A_{14})\}$ and therefore, $Y_{P2}(A_{46}) \geq \{Y_{P2}(A_9), Y_{P2}(A_{10}), Y_{P2}(A_{11}), Y_{P2}(A_{12}), Y_{P2}(A_{13}), Y_{P2}(A_{14})\}$. As a result, when $MPR_{P2} > Y_{P2}(A_{46})$ policies $A_{9}$ through $A_{14}$ cannot be optimal. The decision maker can eliminate the critical ratios $\alpha_1^{P1,P2}$, $\gamma_2^{M1,P1}$, $\lambda_2^{M1,M2}$, $\gamma_3^{M1,P1}$, and $\lambda_3^{M1,M2}$ from consideration. The decision is now limited to the comparison of the critical ratios $\alpha_2^{P1,P2}$, and $\alpha_3^{P1,P2}$ in states 2, and 3, and $\lambda_4^{M1,M2}$ in state 4.

**Proof of Proposition 7:** Let us begin the proof with a monotone policy. Suppose product $P1$ is manufactured in states 1 through $j-1$, and maintenance action $M1$ is performed in states $j+1$ through $N$. Let us use define the following two policies: $A_j = [a_1 = P1, \ldots, a_{j-1} = P1, a_j = M1, \ldots, a_N = M1]$ and $A_{j+1} = [a_1 = P1, \ldots, a_j = P1, a_{j+1} = M1, \ldots, a_N = M1]$. When the firm considers the switch from maintenance to production in state $j$, the base policy is $A_j$, and when it considers the switch in state $j+1$, the base policy is $A_{j+1}$. The ratio of the profit earned from manufacturing $P1$, $r_{j,P_1}$ (which can be defined as decreasing in $j$, corresponding to lower profits in deteriorated states), to the maintenance cost $c_{j,M1}$ (which can be defined as increasing in $j$, corresponding to higher maintenance costs in deteriorated states) is useful for the proof. Let us begin with considering the case when the ratio $r_{j+1,P_1}/c_{j+1,M1}$ is decreasing in state $j$; thus, $r_{j+1,P_1}/c_{j+1,M1} - r_{j,P_1}/c_{j,M1} < 0$. The change in the critical ratio in (10), which can be expressed as $\gamma_j^{M1,P1} - \gamma_{j+1}^{M1,P1}$, can be compared with the change in $r_{j+1,P_1}/c_{j+1,M1} - r_{j,P_1}/c_{j,M1}$. Specifically, if the decrease in $\gamma_j^{M1,P1}$ is higher than the change in $r_{j+1,P_1}/c_{j+1,M1} - r_{j,P_1}/c_{j,M1}$, then the ratio $r_{j+1,P_1}/c_{j+1,M1}$ and $\gamma_j^{M1,P1}$ can intersect and cross signs only once. Therefore, it is sufficient to consider the case when

$$\gamma_j^{M1,P1} - \gamma_{j+1}^{M1,P1} = \frac{r_{j+1,P_1}}{c_{j+1,M1}} - \frac{r_j}{c_j} \geq 0.$$
Observe that from the definition of \( \hat{\Pi}(A_{j+1}) \) and \( \hat{\Pi}(A_j) \), using the above three properties, the above inequality can be expressed as follows:

\[
\sum_{i=1}^{j} \left[ -r_{i,p_1} \left( \frac{\hat{\Pi}(A_{j+1}) - \Delta_{i,j}^{M_1} P_1(A_{j+1})}{c_{i,j,M_1} \hat{\Pi}(A_{j+1})} \right) \right] + \sum_{i=j+2}^{N} \left[ \frac{\hat{\Pi}(A_{j+1})}{c_{i,j,M_1} \hat{\Pi}(A_{j+1})} \right] \geq 0
\]

The following properties can be observed from their definitions:

1. \( \hat{\Pi}(A_{j+1}) = \hat{\Pi}(A_j) - \Delta_{i,j}^{M_1} P_1(A_j) \) for all \( i < j \); thus, \( \hat{\Pi}(A_{j+1}) < \hat{\Pi}(A_j) \) for all \( i < j \).

2. \( \hat{\Pi}(A_{j+1}) = \hat{\Pi}(A_j) \) for all \( i = j \).

3. \( \hat{\Pi}(A_{j+1}) = \hat{\Pi}(A_j) + \Delta_{i,j}^{M_1} P_1(A_j) \) for all \( i > j \); \( \hat{\Pi}(A_{j+1}) > \hat{\Pi}(A_j) \) for all \( i > j \).

Using the above three properties, the above inequality can be expressed as follows:

\[
\sum_{i=1}^{j} \left[ \left( \frac{-r_{i,p_1} + r_{i,p_1} P_1(A_{j+1})}{c_{i,j,M_1} \hat{\Pi}(A_{j+1})} \right) \hat{\Pi}(A_{j+1}) \right] + \sum_{i=j+2}^{N} \left[ \frac{c_{i,j,M_1} \hat{\Pi}(A_j)}{c_{i,j,M_1} \hat{\Pi}(A_{j+1})} \hat{\Pi}(A_{j+1}) \right] \geq 0
\]

Observe that

\[
\sum_{i=1}^{j+1} \left( \frac{-r_{i,p_1} + r_{i,p_1} P_1(A_{j+1})}{c_{i,j+1,M_1} \hat{\Pi}(A_{j+1})} \right) \hat{\Pi}(A_{j+1}) + \sum_{i=j+2}^{N} \left( \frac{c_{i,j,M_1} + r_{i,j+1} M_1 E V (A_{j+1})}{c_{i,j,M_1} \hat{\Pi}(A_{j+1})} \right) \hat{\Pi}(A_{j+1}) = 0
\]

from the definition of \( EV(A_{j+1}) \). Therefore, the above inequality can be simplified to the following:
Adding and subtracting the term \( (r_{j+1,P1} - \tau_{j+1,P1} EV(A_{j+1})) \tilde{\Pi}_{j+1}(A_{j+1}) \) into the left-hand side of the inequality provides:

\[
\sum_{i=1}^{j} \tau_{i,p1} \tilde{\Pi}_{i}(A_{j+1}) + \sum_{i=j+2}^{N} c_{i,m1} \tilde{\Pi}_{i}(A_{j+1}) \geq 0
\]

The above condition is stated as condition 1 in (11), and can be satisfied easily. Also note that
The above inequality is satisfied when

$$\tau_{j+1,P1} + c_{j+1,M1} \leq \left[ \sum_{i=1}^{j+1} \tau_{i,P1} \left( \frac{\bar{h}_i(A_{j+1})}{\bar{p}_{j+1}(A_{j+1})} \right) + \sum_{i=j+2}^{N} \tau_{i,M1} \left( \frac{\bar{h}_i(A_{j+1})}{\bar{p}_{j+1}(A_{j+1})} \right) \right] (EV(A_{j+1}) - EV(A_j))$$

This corresponds to condition 2 in (12) in the proposition. Thus, when conditions 1 and 2 are satisfied,

$$\begin{align*}
\sum_{i=1}^{j+1} \left[ (r_{i,P1} - \tau_{i,P1} EV(A_{j+1})) \left( \frac{\Delta_{j+1,M1}^{j+1}(A_{j+1})}{\bar{p}_{j+1}(A_{j+1})} \right) \right] \\
+ \sum_{i=j+2}^{N} \left[ (c_i,M1 + \tau_{i,M1} EV(A_{j+1})) \left( \frac{\Delta_{j+1,M1}^{j+1}(A_{j+1})}{\bar{h}_i(A_{j+1})} \right) \right] \\
- \sum_{i=1}^{j+1} \left[ (c_{i+1,M1} + \tau_{i+1,M1} EV(A_{j+1})) \left( \frac{\bar{h}_i(A_{j+1})}{\bar{p}_{j+1}(A_{j+1})} \right) \right] \\
+ \sum_{i=j+2}^{N} \left[ (c_i,M1 + \tau_{i,M1} EV(A_{j+1})) \left( \frac{\bar{h}_i(A_{j+1})}{\bar{p}_{j+1}(A_{j+1})} \right) \right] \geq 0
\end{align*}$$

and

$$\left[ \gamma_{j+1}^{M1,P1} - \gamma_{j}^{M1,P1} \right] - \left[ \frac{\tau_{j+1,P1}}{\tau_{j,M1}} - \frac{r_{j,P1}}{\tau_{j,M1}} \right] \geq 0.$$

**Proof of Proposition 8:** a) In (14), when $\delta_{j,M1,M2}^{P1,P2} < \frac{\tau_{j,M2}}{\tau_{j,M1}}$, the firm has $\left( \tau_{j,P2} - \tau_{j,P1} \delta_{j,P1,P2} \right) > 0$. Because $EV(A_n) > 0$, the second term is ensured to be positive. When $\left( \frac{\tau_{j,P2} - \tau_{j,P1} \delta_{j,P1,P2}}{\tau_{j,P1}} \right)$ is increasing, both terms in (14) increase, resulting in increasing values of the critical ratio. b) The same approach can be used to prove the decreasing behavior of (14).

**Proof of Proposition 9:** In Proposition 8.a), it is shown that $\alpha_{j}^{P1,P2}$ is increasing when the conditions 2, 3, and 4 are satisfied. If $\frac{\tau_{j,P2}}{\tau_{j,M1}}$ is increasing in $j$, as the first condition states, the relative values of $\frac{\tau_{j,P2}}{\tau_{j,M1}}$ and $\alpha_{j}^{P1,P2}$ can switch only once. Specifically, if $\frac{\tau_{j,P2}}{\tau_{j,M1}} > \alpha_{j}^{P1,P2}$, product $P2$ is preferred. As $\frac{\tau_{j,P2}}{\tau_{j,M1}}$ decreases in state $j$, the firm can have $\frac{\tau_{j,P2}}{\tau_{j,M1}} < \alpha_{j}^{P1,P2}$ in a state $j$, and switch to manufacturing $P1$. Because of the decreasing behavior of $\frac{\tau_{j,P2}}{\tau_{j,M1}}$, the firm would have $\frac{\tau_{j,P2}}{\tau_{j,M1}} < \alpha_{j}^{P1,P2}$ in all states $i > j$.

**Proof of Proposition 10:** a) In (15), when $\delta_{j,M1,M2}^{M1,M2} > \frac{\tau_{j,M2}}{\tau_{j,M1}}$, the firm has $\left( -\tau_{j,M2} + \tau_{j,M1} \delta_{j,M1,M2}^{M1,M2} \right) > 0$. Because $EV(A_n) > 0$, the second term is ensured to be positive. When $\left( \frac{-\tau_{j,M2} + \tau_{j,M1} \delta_{j,M1,M2}^{M1,M2}}{\tau_{j,M1}} \right)$ is increasing, both terms in (15) increase, resulting in increasing values of the critical ratio. The same approach can be used to prove the decreasing behavior of (15).

**Proof of Proposition 11:** In Proposition 10.a), it is shown that $\lambda_{j}^{M1,M2}$ is increasing when the conditions 2, 3, and 4 are satisfied. If $\frac{\tau_{j,M2}}{\tau_{j,M1}}$ is decreasing in $j$, as the first condition states, the relative values of $\frac{\tau_{j,M2}}{\tau_{j,M1}}$ and $\lambda_{j}^{M1,M2}$ can switch only once. Specifically, if $\frac{\tau_{j,M2}}{\tau_{j,M1}} > \lambda_{j}^{M1,M2}$, maintenance action $M1$ is preferred. As $\frac{\tau_{j,M2}}{\tau_{j,M1}}$ decreases in state $j$, the firm can have $\frac{\tau_{j,M2}}{\tau_{j,M1}} < \lambda_{j}^{M1,M2}$ in a state $j$, and switch to maintenance action $M2$. Because of the decreasing behavior of $\frac{\tau_{j,M2}}{\tau_{j,M1}}$, the firm would have $\frac{\tau_{j,M2}}{\tau_{j,M1}} < \lambda_{j}^{M1,M2}$ in all states $i > j$.

**Example 1:** This example highlights the impact of the minimum production requirements on the optimal policy choice and the critical ratios used in determining the optimal solution. Consider the scenario in which in the two products earn a per unit profit of $\rho_{P1} = 15$ and $\rho_{P2} = 45$, respectively. As the process deteriorates, the number of defective units produced increases. The yield for $P1$ in various states is $y_{1,P1} = 100$, $y_{2,P1} = 90$, $y_{3,P1} = 80$, and for $P2$ it is $y_{1,P2} = 95$, $y_{2,P2} = 65$, $y_{3,P2} = 60$. The revenue earned for each product in each state is described as $r_{i,a_i} = \rho_{P1} \times y_{i,a_i}$, and therefore, $r_{1,P1} = 1$, $r_{2,P1} = 1$, $r_{3,P1} = 1$, $r_{1,P2} = 4$, $r_{2,P2} = 2$, $r_{3,P2} = 2$, $r_{4,P2} = 2$, $r_{5,P2} = 2$, $r_{6,P2} = 2$.
for $P_2$. The expected processing times for each product are $\tau_{i,P_1} = 1$ and $\tau_{i,P_2} = 2$ in states $i = 1,2,3$. One of the trade-offs between the two products is that $P_2$ takes a longer expected time to produce but brings a higher profit. The maintenance costs are as follows: $c_{i,M_1} = $1,300 for a minor maintenance action and $c_{i,M_2} = $2,600 for a major maintenance activity in states $i = 2,3,4$. The expected time for each maintenance action is $\tau_{i,M_1} = 1$ and $\tau_{i,M_2} = 2$ in states $i = 2,3,4$. Therefore, a major maintenance action is twice long in its expected processing time and twice expensive compared with a minor maintenance action. The probability transition matrix describing the deterioration from manufacturing actions and the improvement from maintenance actions are as follows:

$$
[P^P_{ij}] = \begin{bmatrix}
0.70 & 0.20 & 0.05 & 0.05 \\
0 & 0.70 & 0.25 & 0.05 \\
0 & 0 & 0.70 & 0.30 \\
0 & 0 & 0 & 1.00
\end{bmatrix}
\quad
[P^M_{ij}] = \begin{bmatrix}
0 & 0.50 & 0.20 & 0.10 \\
0 & 0.50 & 0.25 & 0.25 \\
0 & 0 & 0.50 & 0.50 \\
0 & 0 & 0 & 1.00
\end{bmatrix}
$$

$$
[P^{M_1}_{ij}] = \begin{bmatrix}
1.00 & 0 & 0 & 0 \\
0.50 & 0.50 & 0 & 0 \\
0.25 & 0.25 & 0.50 & 0 \\
0.10 & 0.20 & 0.20 & 0.50
\end{bmatrix}
\quad
[P^{M_2}_{ij}] = \begin{bmatrix}
1.00 & 0 & 0 & 0 \\
0.75 & 0.25 & 0 & 0 \\
0.35 & 0.35 & 0.30 & 0 \\
0.30 & 0.20 & 0.20 & 0.30
\end{bmatrix}
$$

If the decision maker ignores the minimum production constraints in (9), then all 64 policies listed in Table 1 are candidates for the optimal solution. It can be verified that in the first three states that the ratio of profits exceed the production-related critical ratios (i.e., $\frac{c_{i,P_2}}{\tau_{i,P_2}} > \alpha_i^{P_1,P_2}$ for $i = 1,2,3$) recommending the manufacturing of product $P_2$, and the ratio of maintenance costs exceeds the maintenance-related critical ratio (i.e., $\frac{c_{i,M_2}}{\tau_{i,M_2}} > \lambda_i^{M_1,M_2}$) employing minor maintenance in the worst state. As a result, in the absence of minimum production requirements, the optimal policy is $A_{15} = [P_2, P_2, P_2, M_1]$ with the expected profit of $EV(A_{15}) = $763.95. According to $A_{15}$, however, the firm should manufacture only product $P_2$ with the expected production of $Y_{P_2}(A_{15}) = 24.579$, without any production of $P_1$, i.e., $Y_{P_1}(A_{15}) = 0$.

The firm may have prior commitments to its customers, however, so it might be necessary to produce both products. Let us now focus on the 28 policies listed in Table 2, but continue to ignore the minimum production requirements in (9). In this variant of the problem, the least costly switch takes place in the second state, therefore, policy $A_{11} = [P_2, P_1, P_2, M_1]$ becomes the new optimal solution with its expected profit of $EV(A_{11}) = $760.35. The change in the policy is due to the fact the firm’s production yield of $P_2$ in state 2 shows a significant drop (from 95 in state 1 to 65 in state 2). As a result, $P_2$ provides a relatively lower profit in proportion than it does in other states compared with the profits generated from $P_1$. The expected production per unit time from policy $A_{11}$ is $Y_{P_1}(A_{11}) = 18.182$ units of $P_1$ and $Y_{P_2}(A_{11}) = 18.131$ units of $P_2$, and are both positive.

Let us now incorporate the minimum production requirements into the problem. Semiconductor manufacturers typically have significant commitments to provide downstream electronics manufacturers with their mature products such as $P_1$. For standard-technology products, the market is established better than the high-end products. So, our example features a higher requirement for $P_1$ than for $P_2$. A similar example can be produced for the opposite case without changing the structural results.

The example considers the minimum production requirements of $MPR_{P_1} = 36$ units of $P_1$ and $MPR_{P_2} = 6$ units of $P_2$ per unit time for the constraint in (9). There are only three policies that satisfy this constraint: $A_6 = [P_1, P_2, P_1, M_2]$, $A_{10} = [P_2, P_1, P_1, M_2]$ and $A_{35} = [P_1, M_1, P_2, M_1]$. The optimal policy under (9) is $A_{10} = [P_2, P_1, P_1, M_2]$ with its expected profit of $EV(A_{10}) = $464.64. Compared with the optimal policy in the absence...
of a minimum production requirement in (9), the firm loses 39.18% \(\frac{\text{\$763.95-\text{-\$464.64}}}{\text{\$763.95 \times 100\%}}\) of its profits from \(A_{15}\) (the pure product policy of \(P2\)) and 38.89% \(\frac{\text{\$760.35-\text{-\$464.64}}}{\text{\$763.95 \times 100\%}}\) of its profits from \(A_{11}\).

Expected production under policy \(A_{10}\) is \(Y_{P1}(A_{10}) = 40.571\) units of \(P1\) and \(Y_{P2}(A_{10}) = 9.161\) units of \(P2\). Note that \(Y_{P1}(A_{10}) = 40.571 > MPR_{P1} = 36\) and \(Y_{P2}(A_{10}) = 9.161 > MPR_{P2} = 6\). It can be concluded that satisfying these minimum production requirements costs the firm approximately 39% of its profits in this example. Because of its bigger requirement, the firm has to produce \(P1\) more frequently under (9). This is accomplished in state 3 based on a similar reason. Once again, the drop in the yield of product \(P2\) in state 3 (60 units compared with 95 in state 1) makes the choice of producing \(P1\) in state 3 a better alternative than other states. Moreover, the optimal maintenance decision in state 4 is \(M2\). Utilizing the minor maintenance action \(M1\) reduces the expected production to 5.182 < 6 and making (9) infeasible. This exemplifies how the critical ratio \(\lambda_{4}^{M1,M2}\) can be eliminated from consideration due to minimum production requirements in (9).

Proposition 5 has already established the conditions for increasing the throughput of both products. In this proposition, it is shown that maintenance can play a strategic role in increasing the expected production of a product. The above comparison highlights the impact of replacing a maintenance action with a production action in a deteriorated state in order to increase the throughput of a high demand product. In the next example, we demonstrate how the firm can increase its expected production by employing a major maintenance action, rather than minor maintenance.

Consider policy \(A_{21} = [P1, P2, M1, M1]\) with its expected production of \(Y_{P1}(A_{21}) = 20.548 < MPR_{P1} = 36\) and \(Y_{P2}(A_{21}) = 15.137 > MPR_{P2} = 6\). We next show how switching to a major maintenance action from a minor maintenance action can increase the throughput. Consider now policy \(A_{24} = [P1, P2, M2, M2]\) with same set of production decisions but with a major maintenance action in states 3 and 4. Policy \(A_{24}\) leads to the expected production of \(Y_{P1}(A_{24}) = 21.428 < MPR_{P1} = 36\) and \(Y_{P2}(A_{24}) = 12.809.137 < MPR_{P2} = 6\). While the throughput for \(P1\) increases with this policy, throughput for \(P2\) decreases. The reason for a small increase in \(P1\) and a reduction in \(P2\) is due to the long expected processing time of maintenance action \(M2\). While employing a major maintenance action can increase the throughput of a product, it cannot always guarantee an increase due to possibly lengthy expected processing times.

The firm can also increase throughput by moving its maintenance to an earlier state. This can be demonstrated by comparing policies \(A_{21} = [P1, P2, M1, M1]\) and \(A_{35} = [P1, M1, P2, M1]\). While both policies have the same number of production and maintenance actions, manufacturing of \(P2\) moves from state 2 to 3 and maintenance \(M1\) moves from state 3 to state 2. Recall that policy \(A_{21} = [P1, P2, M1, M1]\) brings an expected production of \(Y_{P1}(A_{21}) = 20.548 < MPR_{P1} = 36\) and \(Y_{P2}(A_{21}) = 15.137 > MPR_{P2} = 6\), and therefore, is an infeasible policy. Policy \(A_{35} = [P1, M1, P2, M1]\), however, leads to the expected production of \(Y_{P1}(A_{35}) = 42.857 > MPR_{P1} = 36\) and \(Y_{P2}(A_{35}) = MPR_{P2} = 6\), and satisfies both of the minimum production requirement constraints. Policy \(A_{35}\) increases the throughput of \(P1\) manufactured in state 1 by increasing the frequency of the process being in this state, i.e., by increasing the steady-state probability. While this policy reduces the throughput for \(P2\), it makes both products satisfy the minimum production requirement in (9). This example shows that the firm can increase the throughput of its products manufactured in better states by moving its maintenance action to an earlier state, i.e., before the process gets extremely deteriorated. Thus, it is not just the frequency of maintenance that influences the firm’s throughput;
the timing of maintenance is as influential in the system’s throughput. In conclusion, through this example, we have demonstrated that the firm can increase the throughput of its products by 1) switching from maintenance to production actions in deteriorated states, 2) employing a major maintenance action rather than minor maintenance, 3) moving its maintenance actions to better states. All three alternatives rely on the changes in the expected processing times.

Let us next present the effectiveness of Proposition 6 under the minimum production requirements in (9). The expected production for products $P1$ and $P2$ using policy $A_{42}$ is $Y_{P1}(A_{42}) = 17.391 < MPR_{P1} = 36$ and $Y_{P2}(A_{42}) = 19.361 > MPR_{P2} = 6$. According to part a) of Proposition 6, the two conditions are not satisfied simultaneously, and therefore, Group 3 policies cannot be eliminated from consideration. This is true because policy $A_{35} = [P1, M1, P2, M1]$ is a Group 3 policy and is one of the candidate solutions. Next, we investigate part b) of Proposition 6. Note that $Y_{P1}(A_{24}) = 21.420 < MPR_{P1} = 36$ and $Y_{P2}(A_{28}) = 13.238 > MPR_{P2} = 6$. These imply that the conditions in part b) of the proposition are met, and no policy in Group 2 can be optimal as reflected in the set of potentially optimal policies. The consequence of this result is the following: The firm must produce in state 3. As a result, the critical ratios $\gamma_{3}^{M1,P1}$ and $\lambda_{3}^{M1,M2}$ for state 3 can be eliminated from consideration, leaving the comparison to only $\alpha_{3}^{P1,P2}$. Parts c) and d) of the proposition utilize the same conditions: $Y_{P1}(A_{40}) = 41.67 > MPR_{P1} = 36$ and $Y_{P2}(A_{46}) = 18.855 > MPR_{P2} = 6$. These imply that conditions in part c) are not met, so not all policies from Groups 2 and 3 can be eliminated. Once again, policy $A_{35} = [P1, M1, P2, M1]$ is a Group 3 policy and is one of the candidate solutions, confirming the conclusion of the proposition. Part d) has the same conclusion as the firm cannot eliminate Group 1 policies as two of the three potentially optimal policies, $A_6 = [P1, P2, P2, M2]$ and $A_{10} = [P2, P1, P1, M2]$ belong to Group 1. In sum, the conclusions from Proposition 6 are consistent with our findings. Moreover, from a critical ratio perspective, the firm has to utilize critical ratios $\alpha_{1}^{P1,P2}$ in state 1, $\alpha_{2}^{P1,P2}$, $\gamma_{2}^{M1,P1}$, $\lambda_{2}^{M1,M2}$ in state 2, $\alpha_{3}^{P1,P2}$ in state 3. With this example, we have demonstrated how the minimum production requirements influence the critical ratios and how they help the decision maker reduce the number of comparisons in determining the optimal policy.

**Example 2:** To highlight the monotonicity conditions, this example is framed as a three-state problem. The firm’s profit from manufacturing $P1$ and $P2$ in states 1 and 2 are $[r_1, P1, r_2, P1] = [8100, 750]$ and $[r_1, P2, r_2, P2] = [8150, 1000]$. Notice that the profit for each manufacturing action is decreasing in state. The maintenance cost of $M1$ and $M2$ in states 2 and 3 are as follows: $[c_{2,M1}, c_{3,M1}] = [625, 780]$ and $[c_{2,M2}, c_{3,M2}] = [1000, 1000]$. The expected processing times of these four actions do not change with state: $[\tau_{1,P1}, \tau_{1,P2}, \tau_{1,M1}, \tau_{1,M2}] = [1, 2.5, 1, 2]$ for all $i$. The machine state transition probabilities for the production actions are:

$$
[p_{ij}^{P1}] = \begin{bmatrix}
0.75 & 0.05 & 0.2 \\
0 & 0.7 & 0.3 \\
0 & 0 & 1
\end{bmatrix} \quad \quad
[p_{ij}^{P2}] = \begin{bmatrix}
0.5 & 0.25 & 0.25 \\
0 & 0.4 & 0.6 \\
0 & 0 & 1
\end{bmatrix},
$$

and the machine state transition probabilities for the maintenance actions are:

$$
[p_{ij}^{M1}] = \begin{bmatrix}
1 & 0 & 0 \\
0.5 & 0.5 & 0 \\
0.05 & 0.45 & 0.5
\end{bmatrix} \quad \quad
[p_{ij}^{M2}] = \begin{bmatrix}
1 & 0 & 0 \\
0.8 & 0.2 & 0 \\
0.25 & 0.5 & 0.25
\end{bmatrix}.
$$

Note that the ratio of deterioration probabilities in states 1 and 2 are: $\delta_{1}^{P1,P2} = \frac{1-p_{22}^{P1}}{1-p_{22}^{P2}} = \frac{0.50}{0.25} = 2.0$ and $\delta_{2}^{P1,P2} = \frac{1-p_{22}^{P1}}{1-p_{22}^{P2}} = \frac{0.60}{0.30} = 2.0$. First, we demonstrate how our critical ratios can determine the optimal policy. Let us consider the
base policy $A_5 = [P1, P1, M2]$ with its expected value $EV(A_5) = $284.50. We first calculate the production-related critical ratio in state 2: $\delta_{1P1}^{P1,P2} = \delta_{1}^{P1,P2} + \left( \frac{EV(A_5)}{\tau_{2,P1}} \right) \left( \tau_{2,P2} - \tau_{2,P2} \delta_{2}^{P1,P2} \right) = 2 + \left( \frac{284.50}{1000} \right) (2.5 - 1 \times 2) = 2.19. The ratio of production profits in state 2 is lower than the critical ratio, i.e., $\frac{r_{1,P2}}{r_{1,P1}} = \frac{4750}{4750} = 1.33 < \alpha_{1P1}^{P1,P2} = 2.19$, and therefore, $a_2 = P1$. Since P1 is presented in state 2, we continue to use policy $A_5 = [P1, P1, M2]$ as the reference policy. The production-related critical ratio for state 1 is: $\alpha_{1P1}^{P1,P2} = \delta_{1}^{P1,P2} + \left( \frac{EV(A_5)}{\tau_{1,P1}} \right) \left( \tau_{1,P2} - \tau_{1,P2} \delta_{2}^{P1,P2} \right) = 2 + \left( \frac{284.50}{1000} \right) (2.5 - 1 \times 2) = 2.14. The ratio of production profits in state 1 is lower than the critical ratio, i.e., $\frac{r_{1,P2}}{r_{1,P1}} = \frac{1500}{1000} = 1.5 < \alpha_{1P1}^{P1,P2} = 2.14$, and therefore, $a_1 = P1$. As a result, policy $A_5 = [P1, P1, M2]$ is the optimal solution to this problem. Although the optimal policy is monotone, the conditions reported in Sloan (2008) are not met.

Specifically, condition C3', corresponding to the reward rate superadditivity property, and condition C5’, corresponding to the holding time subadditivity property, are both violated. For the former condition, note that $\frac{r_{1,P1}}{r_{1,P1}} = \frac{1000}{1000} = 1.0$, while $\frac{r_{2,P1}}{r_{2,P1}} = \frac{500}{500} = 1.0$, which is less than 1. For the latter condition, note that $\frac{r_{1,P1}}{r_{1,P1}} = \frac{1000}{1000} = 1.0$, while $\frac{r_{2,P1}}{r_{2,P1}} = \frac{500}{500} = 1.0$, which is not equal to 0. As a result, all four conditions of Proposition 9 are satisfied ensuring that the optimal policy is monotone, whereas the conditions in Sloan (2008) are not met. In addition, we have demonstrated how the critical ratios are used to determine the optimal solution.

**Example 3:** To highlight the monotonicity conditions, this example is framed as a three-state problem. The firm’s profit from manufacturing $P1$ and $P2$ in states 1 and 2 are $[r_{1,P1}, r_{2,P1}] = [350, 120]$ and $[r_{1,P2}, r_{2,P2}] = [500, 100]$. Notice that the profit for each manufacturing action is decreasing in state. The maintenance cost of $M1$ and $M2$ in states 2 and 3 are as follows: $[c_{2,M1}, c_{3,M1}] = [200, 300]$ and $[c_{2,M2}, c_{3,M2}] = [375, 375]$. Note that the maintenance cost for $M1$ is increasing in state whereas it remains the same for $M2$. The processing times of these four actions do not change with state and are equal to: $[\tau_{1,P1}, \tau_{1,P2}, \tau_{1,M1}, \tau_{1,M2}] = [1, 2, 1, 2]$. The machine state transition probabilities for the production actions are:

$$
[p_{ij}^P] = \begin{bmatrix}
0.5 & 0.25 & 0.25 \\
0 & 0.5 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

and the machine state transition probabilities for the maintenance actions are:

$$
[p_{ij}^{M1}] = \begin{bmatrix}
1 & 0 & 0 \\
0.5 & 0.5 & 0 \\
0.1 & 0.5 & 0.4
\end{bmatrix}
$$

Note that the ratio of improvement probabilities in states 2 and 3 are: $\delta_{2}^{M1,M2} = \frac{1 - \frac{1}{E_{2,P2}}}{1 - \frac{1}{E_{2,P2}}} = \frac{990}{950} = 1.08$ and $\delta_{3}^{M1,M2} = \frac{1 - \frac{1}{E_{3,P2}}}{1 - \frac{1}{E_{3,P2}}} = \frac{950}{950} = 1.00$. First, we demonstrate how our critical ratios can determine the optimal policy. Let us consider the base policy $A_9 = [P1, M1, M1]$ with its expected value $EV(A_9) = $17.90. We first calculate the production-related critical ratio in state 3: $\lambda_{3}^{M1,M2} = \delta_{3}^{M1,M2} + \left( \frac{EV(A_9)}{\tau_{3,M1}} \right) \left( \tau_{3,M2} + \tau_{3,M1} \delta_{3}^{M1,M2} \right) = 1.50 + \left( \frac{17.90}{375} \right) (-2 + 1 \times 1.50) = 1.47. The ratio of maintenance costs in state 3 is lower than the critical ratio, i.e., $\frac{c_{3,M2}}{c_{3,M1}} = = \frac{375}{375} = 1.25 < \lambda_{3}^{M1,M2} = 1.47,$
and therefore, \( a^*_2 = M2 \). Now that \( M2 \) is preferred in state 3, we have the policy \( A_{10} = [P1, M1, M2] \) with its expected value \( EV(A_{10}) = 38 \). In state 2, we compare this policy with policy \( A_{15} = [P1, M2, M2] \). The maintenance-related critical ratio for state 2 is: \( \lambda_{2}^{M1,M2} = \delta_{2}^{M1,M2} + \left( \frac{EV(A_{10})}{c_{2},M1} \right) \left( -\tau_{2,M2} + \tau_{2,M1}\delta_{2}^{M1,M2} \right) = 1.80 + (\frac{38}{200})(-2 + 1 \times 1.80) = 1.76 \). The ratio of maintenance costs in state 2 is greater than the critical ratio, i.e., \( \frac{c_{2},M2}{c_{2},M1} = \frac{375}{200} = 1.875 > \lambda_{2}^{M1,M2} = 1.76 \), and therefore, \( a^*_2 = M1 \). Similarly, we can check the best action in state 1 by comparing policy \( A_{10} = [P1, M1, M2] \) with \( A_{14} = [P2, M1, M2] \). The comparison of the ratio of profits with the critical ratio of state 1 reveals that the optimal action is \( P1 \). As a result, policy \( A_{10} = [P1, M1, M2] \) is the optimal solution to this problem. Expected values of all policies are provided in the Appendix where the monotone policy \( A_{10} \) is proven to be the optimal policy. Although the optimal policy is monotone, the conditions reported in Sloan (2008) are not met. Specifically, Condition 4, corresponding to the subadditive property of the transition probabilities, is violated. For \( l = 2 \) and \( i = 2 \), \( \sum_{j=1}^{N} [p_{ij}^{M2} - p_{ij}^{M1}] = -0.4 \), while for \( l = 2 \) and \( i = 3 \), \( \sum_{j=1}^{N} [p_{ij}^{M2} - p_{ij}^{M1}] = -0.29 \), an increase, which violates condition C4. In contrast, the sufficient conditions in Proposition 11 are satisfied for this problem. Condition 1 is met as the ratio of maintenance costs is decreasing in state: \( \frac{c_{2},M2}{c_{2},M1} = \frac{375}{200} = 1.875 > \frac{c_{2},M2}{c_{2},M1} = \frac{375}{200} = 1.25 \). Condition 2 is satisfied because the ratio of improvement probabilities is decreasing in state: \( \delta_{2}^{M1,M2} = 1.80 > \delta_{3}^{M1,M2} = 1.50 \). The problem complies with Condition 3 because \( \frac{\tau_{2,M2}}{\tau_{2,M1}} = \frac{2}{1} = 2 > \delta_{2}^{M1,M2} = 1.80 \) in state 2 and \( \frac{\tau_{3,M2}}{\tau_{3,M1}} = \frac{2}{1} = 2 > \delta_{3}^{M1,M2} = 1.50 \) in state 3. Condition 4 is also satisfied as \( \frac{-\tau_{j,M2} + \tau_{j,M1}\delta_{j}^{M1,M2}}{c_{j,M2}} \) is decreasing in \( j \): \( \frac{-\tau_{2,M2} + \tau_{2,M1}\delta_{2}^{M1,M2}}{c_{2,M2}} = \frac{-2 + 1 \times 1.80}{200} = -0.001 > \frac{-\tau_{3,M2} + \tau_{3,M1}\delta_{3}^{M1,M2}}{c_{3,M2}} = \frac{-2 + 1 \times 1.80}{200} = -0.00167 \). As a result, all four conditions of Proposition 11 are satisfied ensuring that the optimal policy is monotone, while the conditions in Sloan (2008) failed to do so. Moreover, it is shown that the critical ratios enable the decision maker to determine the optimal solution.