1. (40%) For an LTI system with input $x(t)$ and output $y(t)$ are related by the differential equation: 
\[ \frac{d}{dt} y(t) + 5y(t) = x(t). \]
The system also satisfies the condition of initial rest.

(a) $y(t) = ?$ for input $x(t) = e^{-(1+3)j}u(t)$.

(b) If input $x(t) = e^{-t}\cos(3t)u(t)$, what is the output $y(t)$?

Solution:
Method 1:

\[ \delta(t) \longrightarrow h(t) \longrightarrow y(t) \]

\[ \frac{d}{dt} h(t) + 5h(t) = \delta(t) \text{ (5 points).} \]
When $t \neq 0$, \[ \frac{d}{dt} h(t) + 5h(t) = 0. \]

$h(t) = Ae^{-5t}u(t)$, where $A$ is a constant. (5 points)

\[ \frac{d}{dt} h(t) + 5h(t) = \delta(t) \]
becomes: 
\[ -5Ae^{-5t}u(t) + Ae^{-5t}\delta(t) + 5Ae^{-5t}u(t) = \delta(t). \]
So $A = 1$.

Therefore, $h(t) = e^{-5t}u(t)$.(5 points)

(a) for input $x(t) = e^{-(1+3)j}u(t)$

\[ y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} e^{-(1+3)j}u(t')e^{-5(t-t')}u(t-t')dt' \]

\[ = \int_{0}^{\infty} e^{-(1+3)j}e^{-5(t-t')}dt' \]

\[ = e^{-5t} \int_{0}^{\infty} e^{(4+3)j}dt' \]

\[ = \frac{e^{-5t}}{4+3j} \left[ e^{(4+3)j}t - 1 \right]u(t) \]

(10 points)
(b) for input \( x(t) = e^{-t} \cos(3t)u(t) \)

\[
y(t) = x(t) * h(t) = \int_{-\infty}^{t} e^{-t} \cos(3t')u(t')e^{-5(t-t')}u(t-t')dt'
\]

\[
y(t) = e^{-5t} \int_{0}^{t} e^{4t'} \cos(3t')dt'
\]

\[
y(t) = e^{-5t} \frac{1}{4} \left[ \cos(3t'e^4) \right]_{0}^{t} + e^{-5t} \frac{3}{4} \int_{0}^{t} \sin(3t')e^{4t'}dt'
\]

\[
y(t) = e^{-5t} \frac{1}{5} \left[ \cos(3t)e^{4t} - 1 \right] + e^{-5t} \frac{3}{16} \int_{0}^{t} \sin(3t')e^{4t'}dt'
\]

\[
y(t) = e^{-5t} \frac{1}{4} \left[ \cos(3t)e^{4t} - 1 \right] + e^{-5t} \frac{3}{16} \int_{0}^{t} \sin(3t) e^{4t} - \frac{9}{16} e^{-5t} \int_{0}^{t} \cos(3t')e^{4t'}dt'
\]

(10 points)

\[
y(t) = x(t) * h(t) = e^{-5t} \int_{0}^{t} e^{5t'} \cos(3t')dt'
\]

\[
y(t) = e^{-5t} \frac{4}{25} \left[ \cos(3t)e^{4t} - 1 \right]u(t) + \frac{3}{25} e^{-t} \sin(3t)
\]

\[
y(t) = e^{-t} \frac{1}{25} \left[ 4 \cos(3t) + 3 \sin(3t) \right]u(t) - \frac{4}{25} e^{-5t} u(t)
\]

(5 points)

Method 2:

\[
\frac{d}{dt} y(t) + 5y(t) = x(t), \text{ becomes } j\omega Y(j\omega) + 5Y(j\omega) = X(j\omega) \text{ in frequency domain.} \quad (10 \text{ points})
\]

Therefore, \( H(j\omega) = \frac{1}{5 + j\omega} \). \( (5 \text{ points}) \)

(a) for input \( x(t) = e^{(-1+3j)t}u(t) \),

\[
X(j\omega) = \frac{1}{1-3j+j\omega} \quad (5 \text{ points})
\]

Therefore \( Y(j\omega) = \frac{1}{1-3j+j\omega} \cdot \frac{1}{5+j\omega} = \frac{1}{4+3j} \left( \frac{1}{1-3j+j\omega} - \frac{1}{5+j\omega} \right) \).
Convert it back to time domain: \( y(t) = \frac{1}{4+3j} \left[ e^{(-1+3j)t}u(t) - e^{-5t}u(t) \right]. \) (5 points)

(b) for input \( x(t) = e^{-t}\cos(3t)u(t) \)

\[
X(j\omega) = \frac{1}{2} \left( \frac{1}{1-3j+ j\omega} + \frac{1}{1+3j+ j\omega} \right), \quad (5 \text{ points})
\]

therefore, \( Y(j\omega) = \frac{1}{2} \left( \frac{1}{1-3j+ j\omega} + \frac{1}{1+3j+ j\omega} \right) \frac{1}{5+j\omega}. \) (5 points)

\[
Y(j\omega) = \frac{1}{2} \left( \frac{1}{1-3j+ j\omega} + \frac{1}{1+3j+ j\omega} \right) \frac{1}{5+j\omega}
\]

\[
= \frac{1}{8+6j} \left( \frac{1}{1-3j+ j\omega} - \frac{1}{5+j\omega} \right) + \frac{1}{8-6j} \left( \frac{1}{1+3j+ j\omega} - \frac{1}{5+j\omega} \right)
\]

Convert it back to time domain: \( y(t) = \frac{1}{8+6j} \left[ e^{(-1+3j)t}u(t) - e^{-5t}u(t) \right] + \frac{1}{8-6j} \left[ e^{(-1-3j)t}u(t) - e^{-5t}u(t) \right] \)

\[
y(t) = e^{-t} \left( 4 \cos 3t + 3 \sin 3t \right) u(t) - \frac{4e^{-5t}}{25} u(t) \) (5 points)
2. (30%) For a discrete signal $x[n]$, given:

(1) $x[n]$ is real and even signal.

(2) $x[n]$ has period of $N = 8$ and Fourier coefficients $a_k$.

(3) $a_{11} = 5$.

(4) $\frac{1}{8} \sum_{n=0}^{7} |x[n]|^2 = 50$.

Show that $x[n] = A \cos(Bn + C)$, and specify the numerical values of $A$, $B$, and $C$.

Solution:

Since $x[n]$ is even and real, $a_k = a_{-k}$, $a_{-k} = a_k^*$, $a_k = a_k^*$, i.e. $a_k$ are real numbers. (5 points)

From Parseval’s relation, $\frac{1}{8} \sum_{n=0}^{7} |x[n]|^2 = 50$, $\Rightarrow \sum_{k=0}^{7} |a_k|^2 = 50$. (5 points)

$a_{11} = 5 \Rightarrow a_1 = 5$. (5 points)

$a_k = a_{-k} \Rightarrow a_{-3} = 5$, i.e. $a_5 = 5$. All other $a_k$ are 0. (5 points)

$x[n] = 5e^{j\frac{2\pi}{8}3n} + 5e^{j\frac{2\pi}{8}2n}$

$= 10 \cos\left(\frac{2\pi}{8}3n\right)$ (5 points)

$A = 10$, $B = \frac{2\pi}{8}3$, $C = 0$. (5 points)
3. (30%) Let \( x(t) \) be a signal with Fourier transform \( X(j\omega) \). Suppose:

(1) \( x(t) \) is real and nonnegative.

(2) \( \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 + j\omega)X(j\omega)d\omega = Ae^{-2t}u(t) \), where A is independent of t.

(3) \( \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = 2\pi \).

Determine the closed form expression for \( x(t) \).

Solution:

Suppose \( g(t) = Ae^{-2t}u(t) \), then \( G(j\omega) = \frac{A}{2 + j\omega} \), (5 points)

\[
X(j\omega) = \frac{G(j\omega)}{(1 + j\omega)} = \frac{A}{(1 + j\omega)(2 + j\omega)} , \quad (5 \text{ points})
\]

\[
\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = \int_{-\infty}^{\infty} \frac{|A|^2}{(1 + \omega^2)(4 + \omega^2)} d\omega 
\]

\[
= \frac{|A|^2}{3} \left[ \int_{-\infty}^{\infty} \frac{1}{(1 + \omega^2)} - \frac{1}{(4 + \omega^2)} \right] d\omega 
\]

\[
\int_{-\infty}^{\infty} \frac{1}{1 + \omega^2} d\omega = \int_{-\infty}^{\infty} \frac{1}{1 + \tan^2 \theta} d\tan \theta 
\]

\[
= \int_{-\infty}^{\infty} \frac{\pi}{2} d\theta 
\]

\[
= \pi 
\]

\[
\int_{-\infty}^{\infty} \frac{1}{4 + \omega^2} d\omega = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{1 + \left(\frac{\omega}{2}\right)^2} \frac{d\omega}{2} 
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{1 + \tan^2 \theta} d\tan \theta 
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} \frac{\pi}{2} d\theta 
\]

\[
= \pi/2 
\]
\[
\int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega = \frac{|A|^2}{3} \int_{-\infty}^{\infty} \left[ \frac{1}{1+\omega^2} - \frac{1}{4+\omega^2} \right] \, d\omega
\]
\[
= \frac{|A|^2}{6} \pi, 
\]
\[
X(j\omega) = \sqrt{12}e^{j\phi}\left[ \frac{1}{1+j\omega} - \frac{1}{2+j\omega} \right].\] (5 points)

\[x(t) \text{ is real. Therefore, } x(t) = \sqrt{12}[e^{-t}u(t) - e^{-2t}u(t)],\] (5 points)